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Opgaver til Makroøkonomi 2 (MakØk2)

3. udgave

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Preface

Forord til 3. udgave

Da kurset tidligere var på engelsk og forelæsningsnoterne er på engelsk, er denne opgavesamling, der bygger på de tidligere kurser, også overvejende på engelsk. Rettelser og enkelte nye opgaver er tilføjet.

Hermed en tak til sidste års instruktor, stud.polit. Peter Kjær Kruse-Andersen, for nyttige kommentarer.

August 2014 CG

Preface to the first edition

This is a collection of exercise problems that have been used in recent years in the course Macroeconomics 2 within the Mathematics-Economics Bachelor Program at the University of Copenhagen.

For constructive criticism I thank the instructors Niklas Brønager and Christian Heebøll-Christensen as well as many previous students who have suffered for bad wording and obscurities in earlier versions of the problems. No doubt, it is still possible to find obscurities. Hence, I very much welcome comments and suggestions of any kind relating to these exercises.

September, 2012 Christian Groth

Remarks on notation

For historical reasons, in some of the exercises the "level of technology" (assumed measurable along a single dimension) is denoted A, in others T.

Whether we write $\ln x$ or $\log x$, the *natural* logarithm is understood.

In discrete-time models the time argument of a variable, x, appears always as a subscript, that is, as x_t. In continuous-time models, the time argument of a variable *may* appear as a subscript rather than in the more common form $x(t)$ (this is to save notation).

Chapter A

Technology. Simple long-run models in discrete time

A.1 *Short questions* (answering requires only a few well chosen sentences)

a) Consider an economy where all firms' technology is described by the same neoclassical production function, $Y_i = F(K_i, L_i)$, $i = 1, 2, ..., N$, with decreasing returns to scale everywhere (standard notation). Suppose there is "free entry and exit" and perfect competition in all markets. Then a paradoxical situation arises in that no equilibrium with a finite number of firms (plants) would exist. Explain.

b) As an alternative to decreasing returns to scale at *all* output levels, introductory economics textbooks typically assume that average cost curve of the firm is decreasing at small levels of production and constant or increasing at larger levels of production. Express what this assumption means in terms of "returns to scale".

c) Give some arguments for the presumption that the average cost curve is downward-sloping at small output levels.

d) In many macro models the technology is assumed to have constant returns to scale (CRS) with respect to capital and labor taken together. What does this mean in formal terms?

e) Often the *replication argument* is put forward as a reason to expect that CRS should hold in the real world. What is the replication argument? Do you find the replication argument to be a convincing argument for the assumption of CRS with respect to capital and labor? Why or why not?

f) Does the logic of the replication argument, considered as an argument about a property of technology, depend on the availability of the different inputs.

g) Robert Solow (1956) came up with a subtle replication argument for CRS w.r.t. the rival inputs at the aggregate level. What is this argument?

h) Suppose that for a certain historical period there has been something close to constant returns to scale and perfect competition, but then, after a shift to new technologies in the different industries, increasing returns to scale arise. What is likely to happen to the market form? Why?

A.2 Consider a firm with the production function $Y = AK^{\alpha}L^{\beta}$, where $A > 0$, $0 < \alpha < 1$, $0 < \beta < 1$.

a) Is the production function neoclassical?

b) Find the marginal rate of substitution at a given (K, L).

c) Draw in the same diagram three isoquants and draw the expansion path for the firm, assuming it is cost-minimizing and faces a given factor price ratio.

d) Check whether the four Inada conditions hold for this function?

e) Suppose that instead of $0 < \alpha < 1$ we have $\alpha \geq 1$. Check whether the function is still neoclassical?

A.3 Consider the production function $Y = \alpha L + \beta KL/(K + L)$, where $\alpha > 0$ and $\beta > 0$.

a) Does the function imply constant returns to scale?

b) Is the production function neoclassical? *Hint:* after checking criterion (a) of the definition of a neoclassical production function in Lecture Notes, Section 2.1.1, you may apply claim (iii) of Section 2.1.3 together with your answer to a).

c) Given this production function, is capital an essential production factor? Is labor?

d) If we want to extend the domain of definition of the production function to include $(K, L) = (0, 0)$, how can this be done while maintaining continuity of the function?

A.4 "The Cobb-Douglas production function has the property that under technical progress, it satisfies all three neutrality criteria if it satisfies one of them." True or false? Explain why.

A.5 *Stocks versus flows.* Two basic elements in long-run models are often presented in the following way. The aggregate production function is described by

$$Y_t = F(K_t, L_t, A_t), \tag{*}$$

where Y_t is output (aggregate value added), K_t capital input, L_t labor input, and A_t the "level of technology". The time index t may refer to period t, that is, the time interval $[t, t+1)$, or to a point in time (the beginning of period t), depending on the context. And accumulation of the stock of capital in the economy is described by

$$K_{t+1} - K_t = I_t - \delta K_t, \tag{**}$$

where δ is an (exogenous and constant) rate of (physical) depreciation of capital, $0 \leq \delta \leq 1$. Evolution in employment (assuming full employment) is described by

$$L_{t+1} - L_t = nL_t, \qquad n > -1. \tag{***}$$

In continuous time models the corresponding equations are: (*) combined with

$$\begin{aligned} \dot{K}(t) &\equiv \frac{dK(t)}{dt} = Y(t) - C(t) - \delta K(t), \qquad \delta \geq 0, \\ \dot{L}(t) &\equiv \frac{dL(t)}{dt} = nL(t), \qquad n \text{ "free"}. \end{aligned}$$

a) At the theoretical level, what denominations (dimensions) should be attached to output, capital input, and labor input in a production function?

b) What is the denomination (dimension) attached to K in the accumulation equation?

c) Are there any consistency problems in the notation used in (*) vis-à-vis (**) and in (*) vis-à-vis (***)? Explain.

d) Suggest an interpretation that ensures that there is no consistency problem.

e) Suppose there are two countries. They have the same technology, the same capital stock, the same number of employed workers, and the same number of man-hours per worker per year. Country a does not use shift work, but country b uses shift work, that is, two work teams of the same size and the same number of hours per day. Elaborate the formula (*) so that it can be applied to both countries.

f) Suppose F is a neoclassical production function with CRS w.r.t. K and L. Compare the output levels in the two countries. Comment.

g) In continuous time we write aggregate (real) gross saving as $S(t) \equiv Y(t) - C(t)$. What is the denomination of $S(t)$.

h) In continuous time, does the expression $K(t) + S(t)$ make sense? Why or why not?

i) In discrete time, how can the expression $K_t + S_t$ be meaningfully interpreted?

A.6 The Solow growth model (cf. Blanchard: *Macroeconomics*, 3rd ed., Ch. 11-12) can be set up in the following way (discrete time version). A closed economy is considered. There is an aggregate production function,

$$Y_t = F(K_t, T_t L_t), \tag{1}$$

where F is a neoclassical production function with CRS, Y is output, K is capital input, T is the technology level, and L is the labor input. So TL is effective labor input. It is assumed that

$$T_t = T_0(1+g)^t, \quad \text{where } g \geq 0, \tag{2}$$

$$L_t = L_0(1+n)^t, \quad \text{where } n \geq 0. \tag{3}$$

Aggregate gross saving is assumed proportional to gross aggregate income which, in a closed economy, equals real GDP, Y:

$$S_t = sY_t, \qquad 0 < s < 1. \tag{4}$$

Capital accumulation is described by

$$K_{t+1} = K_t + S_t - \delta K_t, \qquad \text{where } 0 < \delta \leq 1. \tag{5}$$

The symbols g, n, and s represent parameters and the initial values T_0, L_0, and K_0, are given (exogenous) positive numbers.

a) What kind of technical progress is assumed in the model?

b) To get a grasp of the evolution of the economy over time, derive a first-order difference equation in the (effective) capital intensity $\tilde{k} \equiv k/T \equiv K/(TL)$, that is, an equation of the form $\tilde{k}_{t+1} = \varphi(\tilde{k}_t)$.

From now on suppose F is Cobb-Douglas.

c) Construct a "transition diagram" in the $(\tilde{k}_t, \tilde{k}_{t+1})$ plane.

d) Examine whether there exists a unique and asymptotically stable (non-trivial) steady state.

e) There is another kind of diagram that is sometimes (especially in continuous time versions of the model) used to illustrate the dynamics of the economy, namely the "Solow diagram". It is based on writing the difference equation of the model on the form $\tilde{k}_{t+1} - \tilde{k}_t = \left(\psi(\tilde{k}_t) - a\tilde{k}_t\right) / [(1+g)(1+n)/s]$. For the case of the general production function (**??**), find the function $\psi(\tilde{k}_t)$ and the constant a. By drawing the graphs of the functions $\psi(\tilde{k}_t)$ and $a\tilde{k}_t$ in the same diagram, one gets a Solow diagram. Indicate by arrows the resulting evolution of the economy.

A.7 We consider the same economy as that described by (1) - (5) in Problem A.6.

a) Find the long-run growth rate of output per unit of labor, $y \equiv Y/L$.

b) Suppose the economy is in steady state up to and including period $t-1$ such that $\tilde{k}_{t-1} = \bar{\tilde{k}} > 0$ (standard notation). Then, at time t (the beginning of period t) an upward shift in the saving rate occurs. Illustrate by a transition diagram the evolution of the economy from period t onward.

c) Draw the time profile of $\ln y$ in the $(t, \ln y)$ plane.

d) How, if at all, is the level of y affected by the shift in s?

e) How, if at all, is the growth rate of y affected by the shift in s? Here you may have to distinguish between temporary and permanent effects.

f) Explain by words the economic mechanisms behind your results in d) and e).

g) As Solow once said (in a private correspondence with Amartya Sen[1]): "The idea [of the model] is to trace full employment paths, no more." What market form is theoretically capable of generating permanent full employment?

h) Even if we recognize that the Solow model only attempts to trace hypothetical time paths with full employment (or rather employment corresponding to the "natural" or "structural" rate of unemployment), the model has at least one important limitation. What is in your opinion that limitation?

A.8 *A more flexible specification of the technology than the Cobb-Douglas function.* Consider the CES production function[2]

$$Y = A\left[\alpha K^{\beta} + (1-\alpha)L^{\beta}\right]^{1/\beta}, \tag{*}$$

where A, α, and β are parameters satisfying $A > 0$, $0 < \alpha < 1$, and $\beta < 1$, $\beta \neq 0$.

a) Does the production function imply CRS? Why or why not?

b) Show that (*) implies

$$\frac{\partial Y}{\partial K} = \alpha A^{\beta}\left(\frac{Y}{K}\right)^{1-\beta} \quad \text{and} \quad \frac{\partial Y}{\partial L} = (1-\alpha)A^{\beta}\left(\frac{Y}{L}\right)^{1-\beta}.$$

c) Express the marginal rate of substitution of capital for labor in terms of $k \equiv K/L$.

d) In case of an affirmative answer to a), derive the intensive form of the production function.

e) Is the production function neoclassical? *Hint:* a convenient approach is to focus on $\partial Y/\partial K$ expressed in terms of k and consider the cases $\beta < 0$ and $0 < \beta < 1$, separately; next use a certain symmetry visible in (*); finally use your answer to a).

[1] *Growth Economics. Selected Readings*, edited by Amartya Sen, Penguin Books, Middlesex, 1970, p. 24.

[2] CES stands for Constant Elasticity of Substitution. You will meet a CES function also in Problem A.9, B.8, and B.17.

f) Draw a graph of $y \equiv K/L$ as a function of k for the cases $\beta < 0$ and $0 < \beta < 1$, respectively. Comment and compare with a Cobb-Douglas function in intensive form, $y = Ak^{\alpha}$.[3]

g) Write down a CES production function with Harrod-neutral technical progress.

A.9 *A potential source of permanent productivity growth* (this exercise presupposes that f) of Problem A.8 has been solved). Consider a Solow-type growth model, cf. Problem A.6. Suppose the production function is a CES function as in (*) of Problem A.8. Let $\beta \in (0, 1)$, $\alpha^{1/\beta}A > (n + \delta)/s$, and ignore technical progress.

a) Express y/k in terms of k, where $k \equiv K/L$ and $y \equiv Y/L$.

b) For a given k_0, illustrate the dynamic evolution of the economy by a "modified Solow diagram", i.e., a diagram with k on the horizontal axis and sy/k on the vertical axis.

c) Find the asymptotic value of the growth rate of k for $t \to \infty$. Comment.

d) What is the asymptotic value of the growth rate of y for $t \to \infty$.

e) The model displays a feature that may seem paradoxical in view of the absence of technical progress. What is this feature and why is it not paradoxical after all, given the assumptions of the model?

A.10 In the last four decades China has had very high growth in real GDP per capita, cf. Table 1. Answer questions a), b), and c) presupposing that the growth performances of China and the U.S. continue to be like what they have been 1980-2007.

a) How many years does it take for China's GDP per capita to be doubled? You should explain your method.

b) How many years does it take for GDP per capita in the U.S. to be doubled?

c) How long time, reckoned from 2007, will it take for China to catch up with the US in terms of income per capita? You should explain your method.

[3] This function can in fact be shown to be the limiting case of the CES function (in reduced form) for $\beta \to 0$.

d) Do you find it likely that the actual course of events will be (approximately) like that? Why or why not?

Table 1. GDP per capita in USA and China 1980 - 2007 (I$ in 2005 Constant Prices)

country	year	rgdpch
United States	1980	24537.41
United States	2007	42886.92
China	1980	1133.21
China	2007	7868.28

Source: PWT 6.3. Note: For China the Version 2 data series is used.

A.11 An important aspect of macroeconomic analysis is to pose good questions in the sense of questions that are brief, interesting, and manageable. If we set aside an hour or so in one of the last lectures at the end of the semester, what question would you suggest should be discussed?

Chapter B

Technology. Productivity growth. Overlapping generations. Fiscal sustainability

B.1 *Short questions.*

a) Make a list of motives for individual saving. Are some of these motives more in focus in an OLG framework than in a Ramsey framework?

b) Briefly give some hints about how you think a Diamond-style OLG model should be extended to give a more adequate picture of the standard life-cycle pattern of individual saving?

c) In a two-period OLG model assume that the number of young people, L_t, grows at the constant rate n. Derive a formula showing the growth of population $N_t \equiv L_t + L_{t-1}$. Comment.

d) Again, in a two-period OLG model assume that the number of young people, L_t, grows at the constant rate n, but suppose that always half of the young die just before becoming old. Find the growth rate of the population. Comment.

e) In standard long-run models with perfect competition (like Diamond's OLG model or the Ramsey model), the real rate of interest, r_t (i.e., a price on the market for loans), and the real rental rate, $\tilde{r}$, for physical capital (i.e., a price on the market for capital services) may or may not coincide for all t. Give a necessary and sufficient condition that they coincide.

B.2 *Short questions.*

a) What is the *golden rule* capital intensity?

b) A steady-state capital intensity can be in the "dynamically efficient" region or in the "dynamically inefficient" region. What is meant by "dynamically efficient" and "dynamically inefficient"? Give a simple characterization of the two regions.

c) Compare some long-run properties of the Ramsey model with the corresponding long-run properties of the Diamond OLG model. *Hint:* For example, think of the long-run interest rate and/or the possibility of dynamic inefficiency.

d) The First Welfare Theorem states that, given certain conditions, any competitive equilibrium ($\equiv$ Walrasian equilibrium) is Pareto optimal. Give a list of circumstances that each tend to obstruct the needed conditions and thus make the conclusion untrue.

B.3 *A two-period saving problem.* Consider the saving problem of the young in a standard Diamond OLG model (as that in Chapter 3). Let the utility discount rate be denoted ρ and let the period utility function be a CRRA function with parameter θ, i.e.,

$$u(c) = \frac{c^{1-\theta} - 1}{1-\theta}, \qquad \theta > 0.$$

a) Interpret θ. *Hint:* calculate the *relative risk aversion*, $-cu''/u'$.

b) Set up the saving problem and find the saving function of the young. Comment on the role of the real wage, the rate of return, and θ.

c) Write down the intertemporal budget constraint with the human wealth (present discounted value of lifetime labor income) of the young on the right-hand side.

d) In the (c_1, c_2) plane (standard notation), illustrate the substitution and income effects of a rise in r.

We now modify the model and assume the considered individual supplies inelastically one unit of labor in both periods of life.

e) Set up the saving problem.

f) Write down the intertemporal budget constraint with the human wealth of the young on the right-hand side.

g) In the (c_1, c_2) plane, illustrate the substitution, income, and *wealth* effects of a rise in r.

h) As to the wealth effect of a rise in r, compare with the standard Diamond OLG model described in a) - d).

B.4 *Social security and the saving of the young.* We consider a simple extension of the Diamond OLG model in order to compare the effects on aggregate saving and capital accumulation of different systems of public pension provision: a funded pension system, i.e., a saving based system (in Denmark for example the ATP system) and a tax-based system (sometimes called pay-as-you-go pension system), hence unfunded (in Denmark called "folkepension"). These two systems are from now named the *funded system* and the *tax-based system*, respectively. The benchmark case is the standard Diamond model without any public pension system, here called *system 0.* For simplicity, technical progress is ignored.

Let the pension received by an old person in period $t+1$ be called p_{t+1} (p for pension) and let the mandatory (i.e., required by law) contribution of a young person in period t be called τ_t (τ for "tax"). Otherwise, the notation is standard. Then, the pension arrangements are as follows:

$$\text{funded system:} \qquad p_{t+1} = (1 + r_{t+1})\tau_t, \tag{*}$$

$$\text{tax-based system:} \qquad p_{t+1} = (1 + n)\tau_{t+1}. \tag{**}$$

Uncertainty and administrative costs are ignored so that the rate of return, r_{t+1}, in the funded system is the same as that on private saving.

a) What is the relationship between n and the dependency ratio (the number of retired people as a proportion of the number of people in the working age population)? Relate the tax-based system to the problems of financing social security in the "ageing society".

The young, looking ahead, wants to maximize $U_t = u(c_{1t})+(1+\rho)^{-1}u(c_{2t+1})$ subject to two period budget constraints.

b) Write down the two period budget constraints in system 0, the funded system, and the tax-based system, respectively.

Let the saving of the young individual in period t be called s_t. Given k_t and the expected rate of return on saving, a temporary equilibrium in period t, leading to fulfilment of the expectation, implies that the following equations hold under system 0:

$$u'(w_t - s_t) = (1+\rho)^{-1}u'((1+r_{t+1})s_t)(1+r_{t+1}), \tag{1}$$

$$w_t = f(k_t) - k_t f'(k_t) \equiv w(k_t), \tag{2}$$

$$r_{t+1} = f'(k_{t+1}) - \delta \equiv r(k_{t+1}), \tag{3}$$

$$k_{t+1} = \frac{s_t}{1+n} \tag{4}$$

In this system w_t, s_t, r_{t+1}, and k_{t+1} are endogenous variables.

c) Briefly interpret the equations. Describe the causal structure of the system.

d) Write down the analogue four equations describing equilibrium under the funded system. Suppose $0 < \tau_t \leq s_t^0$, where s_t^0 is the solution for s_t under system 0. What is the effect of the funded system on capital formation compared to system 0? (Give a formal argument as well as an intuitive explanation.)

e) Set up a similar formal description of equilibrium under the tax-based system, assuming, for simplicity, $\tau_t = \tau$ for all t, where τ is a positive parameter.

f) Show that the saving s_t of the young individual can be written as a function $s_t = s(w_t, r_{t+1}, \tau)$ and sign $\partial s_t/\partial \tau$. Explain the sign intuitively. Do you expect this partial equilibrium effect to be the same as the general equilibrium effect on s_t of increased τ? Why or why not?

B.5 *Social security and capital accumulation.* In this problem the setup is the same as in Problem B.4 and it is presupposed that you have solved that problem.

a) For the case of the tax-based system derive a formula for the general equilibrium effect of increased τ on k_{t+1}, assuming that $s_r(w(\cdot), r(\cdot), \tau) > (1+n)/f''(\cdot)$ for all relevant (k_t, k_{t+1}). Comment on the sign of the effect. *Hint:* consider the fundamental difference equation of the model.

b) Suppose the economy is in a stable steady state up to period t_0. Then, at the beginning of period t_0, τ is increased a little to $\tau' > \tau$. Illustrate the general equilibrium effect by a phase diagram. Explain the intuition.

c) Derive the effect of increased τ on the steady-state value of k.

d) In addition to the stable steady state assumed to exist, if capital is essential, there is in a tax-based system with fixed τ also at least one unstable steady state below the stable one. Why? *Hint:* consider $w(k_t) - \tau$ for decreasing k_t.

B.6 *Discussion of social security reform.* To be able to answer this problem, solving Problem B.4 and B.5 first may be useful.

In discussions of social security reform, among other things the perspectives a), b), and c) below may be raised. Comment on each of them.

a) Suppose that in order to increase aggregate saving and decrease vulnerability to changing n, the government wants to shift society's mandatory pension system from a tax-based system to a funded system. Why is this not a non-problematic thing to do? *Hint:* there is a "hidden debt" to the currently old generation.

b) The distinction between dynamic efficiency and dynamic inefficiency might have a role to play in the discussion. How?

c) Suppose that in steady state $r > n$. Then, with a constant mandatory contribution τ (set $\tau_t = \tau_{t+1} = \tau$ in (*) and (**) of Problem B.4), it seems that a funded system in some sense pays a higher return on deposits than a tax-based system. How?

d) Can we conclude from this observation that society would gain in efficiency by moving from a tax-based system to a funded system? Why or why not?

B.7 *Social security in a small open economy.* Change the setup for the tax-based pension system in a Diamond OLG framework as described in Exercise B.4 to a small open economy (SOE) setup. Suppose there is perfect mobility of financial capital across borders, but no mobility of labor. Suppose domestic and foreign financial claims are perfect substitutes. Assume there is a given positive constant real interest rate, r, in the world market. The SOE has neoclassical CRS technology and perfect competition in all markets. There is no government debt.

Let national wealth be denoted V_t. Then $V_t \equiv K_t - F_t$, where K_t is aggregate capital and NFD_t is net foreign debt of the SOE (all in real terms). Suppose the mandatory contribution, τ_t, to the pension system is a positive constant, i.e., $\tau_t = \tau > 0$.

a) Find the equilibrium real wage. Comment on your solution. *Hint:* the competitive firms of the SOE maximize profits facing a given and constant capital cost, $r + \delta$; determine the capital intensity they choose.

b) Determine the saving of the young in period t. Comment.

Consider national wealth per worker, $v_t \equiv V_t/L_t$, where L_t is the number of young people in the SOE.

c) In analogy with the equation (4) of Problem B.4, what is the equation satisfied by v_{t+1}? *Hint:* since by assumption there is no government debt, national wealth equals private financial wealth.

d) Derive the general equilibrium effect on v_{t+1} of increased τ. Compare with the general equilibrium effect on v_{t+1} in a closed economy, assuming $s_r > 0$.

e) What is the growth rate of V_t and K_t, respectively?

B.8 *An economy with backward technology and absence of technology growth.* Suppose the production function in Diamond's OLG model is $Y = A(\alpha K^\beta + (1-\alpha)L^\beta)^{1/\beta}$, $A > 0, 0 < \alpha < 1, \beta < 0$, and $A\alpha^{1/\beta} < 1 + n$. a) Given $k \equiv K/L$, find the equilibrium real wage, $w(k)$. b) Show that $w(k) < (1+n)k$ for all $k > 0$. *Hint:* consider the "roof". c) Comment on the implication for the long-run evolution of the economy.

B.9 *Comparing Diamond and Solow.* Set up a Diamond OLG model with constant population growth rate $n \geq 0$ and Harrod-neutral technical progress at a constant rate $g > 0$. Let the period utility function be $u(c) = \ln c$ and let the production function be Cobb-Douglas.

a) Derive the fundamental difference equation of the model.

b) Will the model generate balanced growth in the long run? Why or why not?

c) Write down the fundamental difference equation ("law of motion") for a Solow growth model in discrete time with the same production function, same n and g, a constant saving rate (aggregate saving-income ratio) $\sigma \in (0, 1)$, and a constant capital depreciation rate $\delta \in [0, 1]$.

d) Is there a special case where the two models coincide? Why or why not?

e) Let the period length be 30 years. Suppose the Diamond model's ρ equals $4/3$ ($\approx (1+0.0286)^{30} - 1$). Use your empirical knowledge concerning plausible values of α and δ (the latter, given a 30 years' period length). Assess whether one can from the Diamond model come close to a Solow model.

B.10 *Government, income taxes and capital accumulation.* Consider a Diamond OLG model extended with government. Let G_t denote government spending on goods and services in period t. Assume $G_t = G_0(1+n)^t$, where G_0 is a given positive number and n is the given constant rate of population growth, $n \geq 0$. The government uses G_t to provide free public consumer services, say free broadcasting services or free exhibitions and museum services.

To finance its purchases, the government levies taxes. To begin, assume that only labor income is taxed. Let the labor income tax rate be denoted τ_t. The government budget is balanced every period. Assume that G_0 is "small" enough in relation to K_0 and L_0 (standard notation) such that for all t we have $0 \leq \tau_t < 1$. Let the aggregate production function satisfy the Inada conditions and ignore technical progress.

a) Assume from now that an individual born at time t has the utility function $U(c_{1t}, c_{2t+1}, G_t, G_{t+1}) = \ln c_{1t} + \beta \ln G_t + (1+\rho)^{-1} [\ln c_{2t+1} + \beta \ln G_{t+1}]$, where β is a positive parameter (an indicator of how strongly the public good is desired). List at least three *special* features of this two-goods-two-periods utility function.

b) Derive the saving function of the young.

c) Write down 1) the national accounting equation for the use of output and 2) the government budget constraint. Find the tax rate τ_t, given k_t (standard notation). Derive the fundamental difference equation of the model and illustrate the dynamics in a diagram.

d) Is it true or not true that in the present model, labor income taxes are similar to lump-sum taxes in their effect on the behavior of the young? Why?

Let the aggregate production function be Cobb-Douglas.

e) How does the long-run capital intensity depend on the level of G_0? A qualitative answer based on a phase diagram is enough.

f) Assume that the economy has been in its steady state until period 0. Then there is an unanticipated change in government tax policy so that also capital income is taxed, i.e., from period 0 capital income is taxed at the same proportional rate as labor income, this common rate being denoted σ_t. The path of government expenditure is unchanged and the budget is still balanced. Find the new tax rate, σ_t, for $t = 0, 1, 2, \ldots$. *Hint:* repeat the steps b) and c) in this new situation.

g) Show that the long-run capital intensity will be higher than with pure labor income taxation. Explain in words.

h) Assume that the economy has been in its new steady state for a long time. Then there is an unanticipated change in government tax policy so that *only* capital income is taxed, i.e., from now capital income is taxed at the rate required for a balanced budget, this rate being denoted $\tilde{\sigma}_t$. How is the long-run capital intensity affected by this?

B.11 *Fiscal sustainability.* Consider the government budget in a small open economy. Time is discrete, the period length is one year, and there is no uncertainty. Let g and n be non-negative constants and let

$$\begin{aligned}
Y_t &= Y_0(1+n)^t(1+g)^t = \text{ real GDP,} \\
G_t &= \text{ real government spending on goods and services,} \\
T_t &= \text{ real net tax revenue (= gross tax revenue -- transfer payments),} \\
B_t &= \text{ real public debt at the start of period } t, \\
r &= \text{ world market real interest rate, a constant.}
\end{aligned}$$

Assume that any government budget deficit is exclusively financed by issuing debt (and any budget surplus by redeeming debt).

a) Write down an equation showing how B_{t+1} is determined by values of variables dated t.

Consider a scenario with $B_0 > 0$, $1 + r > (1 + g)(1 + n)$, and $T_t/Y_t = \tau$, a positive constant less than one.

b) Find the maximum constant G/Y which is consistent with fiscal sustainability. *Hint:* the difference equation $x_{t+1} = ax_t + b$, where a and b are constants, $a \neq 1$, has the solution $x_t = (x_0 - x^*)a^t + x^*$, where $x^* = b/(1 - a)$.

B.12 Consider a budget deficit rule saying that $\lambda \cdot 100$ percent of the interest expenses on public nominal debt, D, plus the primary budget deficit must not be above $\alpha \cdot 100$ percent of nominal GDP, PY, where Y is real GDP, growing at a constant rate, $g_Y > 0$, and P is the GDP deflator. So the rule requires that

$$\lambda i D_t + P_t(G_t - T_t) \leq \alpha P_t Y_t, \tag{*}$$

where $\lambda > 0$, $\alpha > 0$, and

$$\begin{aligned}
G_t &= \text{real government spending on goods and services in period } t, \\
T_t &= \text{real net tax revenue in period } t, \\
i &= (1+r)(1+\pi) - 1, \text{ where } r \text{ is the real interest rate}, \\
\pi &= \frac{P_t - P_{t-1}}{P_{t-1}} = \text{the inflation rate, a given non-negative constant.}
\end{aligned}$$

a) Is the deficit rule of the Stability and Growth Pact in the EMU a special case of (*)? Comment.

b) Let $b_t \equiv D_t/(P_{t-1}Y_t)$. Derive the law of motion (difference equation) for b_t, assuming the deficit ceiling is always binding. *Hint:* $\text{GBD}_t = iD_t + P_t(G_t - T_t)$.

Suppose λ is such that $0 < 1 + (1-\lambda)i < (1+\pi)(1+g_Y)$.

c) For an arbitrary $b_0 > 0$, find the time path of b. Briefly comment. *Hint:* the difference equation $x_{t+1} = ax_t + c$, where a and c are constants, $a \neq 1$, has the solution $x_t = (x_0 - x^*)a^t + x^*$, where $x^* = c/(1-a)$.

d) How does a rise in λ affect the long-run debt-income ratio? Comment.

B.13 *A three-period OLG model.* Consider an extension of the Diamond OLG model such that people live for three periods of equal length. For an individual born at time t, let c_{1t}, c_{2t+1} and c_{3t+2} be the consumption in the first period of life ("youth"), the second period of life ("middle age period"), and the third period of life ("retirement period"), respectively. The utility function of a young born at time t is $U(c_{1t}, c_{2t+1}, c_{3t+2}) = \ln c_{1t}$ $+(1+\rho)^{-1} \ln c_{2t+1}$ $+(1+\rho)^{-2} \ln c_{3t+2}$, where $\rho > -1$. Individuals supply inelastically $\bar{x}$ units of labor in the first period of life, one unit of labor in the second period of life, while people don't work in the third period of life. The technology side of the model is as in the two-period Diamond model, people have perfect foresight and all markets are competitive. The rate of population growth is a constant $n > -1$.

a) From now, assume $0 < \bar{x} < 1$. Is this assumption natural? Why or why not?

b) For notational simplicity, disregard for a moment the time indices t, $t+1$, and $t+2$. Set up the optimization problem of the young.

c) Derive the intertemporal budget constraint and find the optimal c_1, c_2, and c_3. Comment.

d) Find the financial wealth, a_2, held by the middle aged individual at the beginning of the second period of life and the financial wealth, a_3, of the old individual at the beginning of the third period of life as functions of the relevant parameters ($a_3 = a_2 + s_2$, where s_2 is the saving of the middle aged individual). Do a_2 and a_3 depend on the rate of return on saving made in the first period? Comment in relation to the assumed logarithmic period utility.

e) Give a list of conditions that could make $a_{2t} < 0$. Comment!

f) Ignoring technical progress, show that the fundamental difference equation of the model is a second order non-linear difference equation.

g) If instead of the log function, a general CRRA utility function is used, the fundamental difference equation would be a third order equation. Guess why.

B.14 *Short- and long-run effects of a technology shock.* Consider a Diamond OLG model for a closed economy. Let the utility discount rate be denoted ρ and let the period utility function be specified as

$$u(c) = \frac{c^{1-\theta} - 1}{1 - \theta}, \qquad \theta > 0.$$

a) In what sense can even the case $\theta = 1$ be interpreted as meaningful?

b) Derive the saving function of the young. Comment.

From now on, assume $u(c) = \ln c$.

c) Let the aggregate production function be a neoclassical production function with CRS and assume Harrod-neutral (i.e., labor-augmenting) technical progress at a constant exogenous rate $g > 0$. Derive the fundamental difference equation of the model. *Hint:* $K_{t+1} = s_t L_t$.

From now, assume that the production function is Cobb-Douglas.

d) Draw a transition diagram illustrating the dynamics of the economy for a given $\tilde{k}_0 > 0$.

e) Is there a unique and asymptotically stable steady state?

f) Are the inferences that can be drawn from the model consistent with Kaldor's stylized facts? Comment.

g) Suppose the economy is in steady state up to and including period $t_0 - 1$. Then, at the shift from period $t_0 - 1$ to period t_0, a positive technology shock occurs such that the technology level, in period t_0 is *above* $1 + g$ times that of period $t_0 - 1$. The rate of technical progress from period $t_0 + i$ to period $t_0 + i + 1$, $i = 0, 1, 2, \ldots$, remains at g, however, and everyone expects it to do so. Illustrate by a transition diagram the evolution of the economy from period $t_0 - 1$ onward.

h) How, if at all, is the real interest rate in the long run affected by the shock?

i) How, if at all, is the real wage in the long run affected by the shock?

j) How, if at all, is the real wage in period t_0 affected by the shock?

B.15 *Short- and long-run effects of increased patience in the Diamond OLG model.* Set up a Diamond OLG model with constant population growth rate $n \geq 0$ and Harrod-neutral technical progress at a constant rate $g > 0$. Let the period utility function be $u(c) = (c^{1-\theta} - 1)/(1 - \theta)$, where $\theta > 0$, and the production function be Cobb-Douglas.

a) Derive the fundamental difference equation of the model. given the real wage w_t and the real interest rate r_{t+1}.

b) Use your general knowledge about the CRRA-Cobb-Douglas case to draw a transition diagram illustrating the dynamics of the economy. Comment on existence, uniqueness, and stability of a steady state of the economy.

Suppose the economy is in steady state up to and including period $t_0 - 1$. Then, at the transition from period $t_0 - 1$ to period t_0 a reduction of the impatience parameter of the young to a lower level occurs. Henceforth the impatience parameter remain at the new lower level for every new generation.

c) Illustrate by a transition diagram how the transition curve is shifted and how the economy evolves from period $t_0 - 1$ onward. (*Hint:* You may use your intuition but it is better if you can also derive your conclusion mathematically. With this aim it may be useful to re-write the steady-state equation such that $\tilde{k}$ only appears through the power function $\tilde{k}^{\alpha-1}$.)

d) Let y denote output per unit of labor. In the $(t, \ln y)$ plane draw the time profile of $\ln y$.

e) How, if at all, is the growth rate of y affected by the shift in the impatience parameter? Here you may have to distinguish between temporary and permanent effects.

f) Explain by words the economic mechanism behind your result in e).

g) What new insight, if any, does the Diamond OLG model add compared to the Solow growth model?

B.16 *A two-period saving problem with CARA utility.*[1] Consider the saving problem of the young in a standard Diamond OLG model (as that in Chapter 3). Let the utility discount rate be denoted ρ and let the period utility function be a CARA function with parameter $\alpha > 0$, i.e.,

$$u(c) = -\alpha^{-1} e^{-\alpha c}.$$

a) Interpret α. *Hint:* calculate the *absolute risk aversion*, $-u''/u'$.

b) Set up the saving problem and find the saving function of the young. Comment on the role of the real wage, the rate of return, and α.

c) Write down the intertemporal budget constraint with the human wealth (present discounted value of lifetime labor income) of the young on the right-hand side.

d) In the (c_1, c_2) plane (standard notation), illustrate the substitution and income effects of a rise in r.

We now modify the model and assume the considered individual supplies inelastically one unit of labor in both periods of life.

e) Set up the saving problem.

[1] Compare with Exercise B.3.

f) Write down the intertemporal budget constraint with the human wealth of the young on the right-hand side.

g) In the (c_1, c_2) plane, illustrate the substitution, income, and *wealth* effects of a rise in r. To fix ideas your illustration may concentrate on the case where the parameters are such that the individual in the initial situation (with the lower r) chooses positive saving in the first period.

B.17 Consider a Diamond OLG model for a closed economy. Let σ be a parameter, $\sigma > 0$. Let the period utility function be specified as

$$u(c) = \begin{cases} \frac{c^{1-1/\sigma}}{1-1/\sigma}, & \text{if } \sigma \neq 1, \\ \ln c, & \text{if } \sigma = 1. \end{cases}$$

a) What is the interpretation of the parameter σ?

b) Let the utility discount rate be denoted ρ, $\rho > -1$. Derive the saving function of the young. In terms of the three Slutsky effects comment on how a rise in the interest rate affects the saving of the young and how the net effect depends on the size of σ.

From now on assume $u(c) = \ln c$.

c) Let the aggregate production function on intensive form be $f(\tilde{k}) = A(\alpha\tilde{k}^{\beta} + 1 - \alpha)^{1/\beta}$, $A > 0$, $0 < \alpha < 1$, $-\infty < \beta < 1$. Here $\tilde{k} \equiv K/(TL)$, where K is capital input, L is labor input, and T is the level of technology, which is assumed to grow at a constant exogenous rate $g > 0$. Given that the number of young grows at the rate n, derive the fundamental difference equation of the model. *Hint:* with Y denoting output, $f(\tilde{k})TL$, the marginal productivity of labor is $\partial Y/\partial L$ $= (1-\alpha)A(\alpha\tilde{k}^{\beta} + 1 - \alpha)^{(1-\beta)/\beta}T$.

It can be shown that $\lim_{\beta \to 0} \left[A(\alpha\tilde{k}^{\beta} + 1 - \alpha)^{1/\beta}\right] = A\tilde{k}^{\alpha}$. From now on assume that $f(\tilde{k}) = A\tilde{k}^{\alpha}$.

d) Draw a transition diagram illustrating the dynamics of the economy. Given $\tilde{k}_0 > 0$, indicate how the economy moves over time. Comment.

e) Suppose the economy is in steady state up to and including period $t_0 - 1 > 0$. Then, at the beginning of period t_0 people become aware that a slowdown in technical progress has occurred to the effect that between period t_0+i and period t_0+i+1, $i = 0, 1, 2, \ldots$, the growth rate of T is at a lower constant level $g' \in (0, g)$. Everyone rightly expects the growth rate of T to remain at this lower level forever. Illustrate by a transition diagram the evolution of the economy from period t_0 onward. Comment.

We now add a government sector to the model. The government has only one activity; the motivation of this activity is that the lowered rate of technical progress has resulted in an equilibrium path which is dynamically inefficient.

f) What is meant by "dynamically inefficient"?

The activity of the government is to issue one-period bonds. A one-period bond issued at the end of period $t-1$ (= beginning of period t) pays the owner a gross return of $1 + r_t$ at the end of period t, where r_t is the real interest rate in the economy. There is no taxation. The budget deficit is financed by issuing additional debt. Let the real value of the government debt at the end of period $t-1$ be denoted B_t.

g) Write down the law of motion (first-order difference equation) for B_t.

h) Why is it that such a policy may help vis-a-vis dynamic inefficiency? *Hint:* Private financial wealth at the beginning of period $t+1$ is now $A_{t+1} = K_{t+1} + B_{t+1}$.

i) Suppose the described policy implies convergence to the golden rule steady state. Will the government then be able to remain solvent? Why or why not?

B.18 Consider a Diamond OLG model for a closed economy. Let the utility discount rate be denoted ρ and let the period utility function be specified as $u(c) = \ln c$.

a) Derive the saving function of the young. Comment.

b) Let the aggregate production function be a neoclassical production function with CRS and assume Harrod-neutral (i.e., "labor-augmenting") technical progress at a constant exogenous rate $g > 0$. Let L_t denote the number of young in period t. Derive the fundamental difference equation of the model.

From now, assume that the production function is the one given in Problem A.3, but extended with Harrod-neutral technical progress at the rate g.

c) Draw a transition diagram illustrating the dynamics of the economy. Make sure that you draw the diagram so as to exhibit consistency with the production function.

d) Given the above information, can we be sure that there exists a unique and globally asymptotically stable steady state? Why or why not?

e) Suppose the economy is in a steady state up to and including period $t_0 > 0$. Then, at the shift from period t_0 to period $t_0 + 1$, a negative technology shock occurs such that the technology level in period $t_0 + 1$ is below $1 + g$ times that of period t_0. The rate of technical progress from period $t_0 + i$ to period $t_0 + i + 1$, $i = 1, 2, \ldots$, remains at g, however, and everyone expects it to do so. Illustrate by a transition diagram the evolution of the economy from period t_0 onward. Comment.

f) Let $k \equiv K/L$. In the $(t, \ln k)$ plane, draw a graph of $\ln k_t$ such that the qualitative features of the time path of $\ln k$ before and after the shock, including the long run, are exhibited. *Hint:* $k \equiv \tilde{k}T$.

g) How, if at all, is the real interest rate in the long run affected by the shock?

h) How, if at all, is the real wage in the long run affected by the shock?

i) How, if at all, is the labor income share of national income in the long run affected by the shock?

j) Explain by words the economic intuition behind your results in h) and i).

k) Are the inferences that can be drawn from the model consistent with Kaldor's stylized facts? Comment.

B.19 Consider a Diamond OLG model for a closed economy. Let the utility discount rate be denoted ρ and the period utility function be specified as $u(c) = \ln c$.

a) Derive the saving function of the young. Comment.

Let the aggregate production function be neoclassical with CRS. Assume there is Harrod-neutral technical progress at a constant exogenous rate $g > 0$. And assume the number, L_t, of young grows at the constant exogenous rate $n > -1$.

b) Derive the fundamental difference equation of the model.

c) Assuming capital is not essential and that at least one steady state exists, draw a transition diagram illustrating the dynamics of the economy. Given an arbitrary $\tilde{k}_0 > 0$ (standard notation), indicate how the economy moves over time. *Hint:* $\lim_{k\to 0}(f(k) - kf'(k)) = f(0)$.

d) Given the assumptions made in c), can we be sure about the number of steady states? Why or why not? *Hint:* $\lim_{k\to\infty}(w(k)/k) = 0$ (standard notation).

e) List five of Kaldor's stylized facts.

f) For each of these facts check whether the inference that can be drawn from the model complies with it.

B.20 Consider a small open economy (SOE) facing a constant real interest rate $r > 0$, given from the world market for financial capital. We ignore business cycle fluctuations and assume that real GDP, Y_t, grows at a constant exogenous rate $g_Y > 0$; we assume $g_Y < r$.

Time is discrete. Further notation is:

$$\begin{aligned} G_t &= \text{real government spending on goods and services,} \\ T_t &= \text{real net tax revenue } (= \text{ gross tax revenue} - \text{transfer payments}), \\ GBD_t &= \text{real government budget deficit,} \\ B_t &= \text{real public debt (all short-term) at the start of period } t. \end{aligned}$$

Assume that any government budget deficit is exclusively financed by issuing debt (and any budget surplus by redeeming debt).

a) Write down two equations showing how GBD_t and B_{t+1}, respectively, are determined by variables indexed by t. Also write down an equation indicating how B_{t+1} is related to GBD_t.

Suppose that $B_0 > 0$ and $G_t = \gamma Y_t$, $t = 0, 1, \ldots$, where $0 < \gamma < 1$. Define the "net tax burden" as $\tau_t \equiv T_t/Y_t$.

b) Find the minimum net tax burden, $\bar{\tau}$, which, if maintained, is consistent with fiscal sustainability. *Hint:* different approaches are possible; one of these focuses on the debt-income ratio and uses the fact that a difference equation $x_{t+1} = ax_t + b$, where a and b are constants, $a \neq 1$, has the solution $x_t = (x_0 - x^*)a^t + x^*$, where $x^* = b/(1-a)$.

c) How does $\bar{\tau}$ depend on r and g_Y, respectively? Comment.

B.21 *The aggregate saving rate.* Consider a Diamond OLG model with a neoclassical CRS production function F so that aggregate output in period t is given as $Y_t = F(K_t, L_t)$, where K_t is capital input and L_t labor input. Suppose the model has a unique and asymptotically stable steady state with capital-labor ratio equal to $k^* > 0$.

a) Let S^N be aggregate net saving. Derive a formula for the aggregate net saving rate (S^N/Y) in the long run in terms of the rate of population growth, the production function on intensive form, and the capital-labor ratio. *Hint:* $S^N = K_{t+1} - K_t \equiv k_{t+1}L_{t+1} - k_tL_t$.

b) Assume now that the rate of population growth is zero. What does the formula tell you about the level of net aggregate saving in steady state in this case? Give the intuition behind the result.

c) The steady state just described may at first sight seem paradoxical if the young are very patient (rate of time preference equal to zero, say). Is it at all possible that the economy can be in the described steady state in this situation? Why or why not?

d) Returning to the general case $(n > -1)$, we now extend the model by adding Harrod-neutral technical progress at the constant rate $g > 0$. Derive a formula for the aggregate saving rate in the long run in the model in this case.

e) In case of zero population growth, is your result from b) still true? Comment.

f) Empirical cross-country studies generally find a positive correlation between the aggregate saving rate and the GDP growth rate. Suggest a causal interpretation consistent with the present model.

g) Briefly suggest a kind of alternative model leading to an opposite interpretation of the causality.

B.22 As in Exercise B.21, consider a Diamond OLG model. A key relationship in the model is the equation (standard notation)

$$K_{t+1} = s_t L_t. \tag{*}$$

a) Explain the equation (*).

b) Is the equation (*) consistent with the national income relationship for a closed economy, $K_{t+1} = K_t + S_t - \delta K_t$, where S_t is aggregate gross saving and δ is the capital depreciation rate? Why or why not?

B.23 *Short questions*

a) A technically feasible path, $\{(c_t, k_t)\}_{t=0}^{\infty}$, in a Diamond OLG economy may be *dynamically inefficient*. What is meant by this?

b) "A given fiscal policy is sustainable if and only if it maintains compliance with the intertemporal budget constraint of the government." What is meant by "the intertemporal budget constraint of the government" and is the statement true or false? Briefly discuss.

B.24 Consider a small open economy (SOE) facing a real interest rate, r_t, given from the world market for financial capital. There is no cross-country mobility of labor. Under "normal circumstances" the following holds:

- aggregate employment, N, is at the "full employment" level, $N = (1 - \bar{u})L$, where $\bar{u}$ is the NAIRU and L is the aggregate labor supply, a given constant;
- real GDP, Y_t, equals its given trend level, $\bar{Y}_t$, which grows at a constant exogenous rate $g_Y > 0$ due to technical progress;
- $r_t = r$, where r is a constant and $r > g_Y$.

Time is discrete. Further notation is:

$$\begin{aligned}
G_t &= \text{real government spending on goods and services,} \\
T_t &= \text{real net tax revenue} = \text{gross tax revenue} - \text{transfer payments,} \\
GBD_t &= \text{real government budget deficit,} \\
B_t &= \text{real public debt (zero coupon one period bonds) start of period } t.
\end{aligned}$$

Assume that any government budget deficit is exclusively financed by issuing debt (and any budget surplus by redeeming debt).

a) Write down two equations showing how GBD_t and B_{t+1}, respectively, are determined by variables indexed by t. Also write down an equation indicating how B_{t+1} is related to GBD_t.

Suppose that $B_0 > 0$ and $G_t = \gamma \bar{Y}_t$, $t = 0, 1, \ldots$, where $0 < \gamma < 1$. Define the "net tax burden" as $\tau_t \equiv T_t/Y_t$.

b) Find the minimum constant net tax burden, $\hat{\tau}$, which is consistent with fiscal sustainability. *Hint:* different approaches are possible; one focuses on the debt-income ratio and uses the fact that a difference equation $x_{t+1} = ax_t + c$, where a and c are constants, $a \neq 1$, has the solution $x_t = (x_0 - x^*)a^t + x^*$, where $x^* = c/(1-a)$.

c) How does $\hat{\tau}$ depend on $r - g_Y$ and $b_0 \equiv B_0/Y_0$, respectively?

d) Suppose the government for some reason (economic or political) can not raise the net tax burden above some threshold value, $\bar{\tau}$, and can not decrease the $G_t/\bar{Y}_t$ below some value, $\bar{\gamma}$. Find the maximum value, $\bar{r}$, of the interest rate consistent with a non-accelerating debt-income ratio. How does $\bar{r}$ depend on b_0? This dependency tells us why for some countries a high debt-income ratio is problematic. Explain.

Now consider an alternative scenario. In period $t = -1$ the SOE is hit by a huge negative demand shock and gets into a substantial recession (henceforth denoted a slump) with Y_{-1} far below $\bar{Y}_{-1}$. In response the government decides an "expansionary fiscal policy" instead of "laissez-faire", where:

- "laissez-faire" means maintaining $G_t = \gamma \bar{Y}_t$, $t = 0, 1, \ldots$;
- "expansionary fiscal policy" entails a discretionary increase in G of size ΔG, beginning in period 0 and maintained during the slump to stimulate economic activity, that is, $G_t = \gamma \bar{Y}_t + \Delta G$, where ΔG is a positive constant.

Let the tax and transfer rules in the economy imply that net tax revenue in period 0 is given by the function $T = T(Y)$; thus, $T_0 = T(Y_0)$. Assume that under the current slump conditions marginal net tax revenue is $T'(Y) = 0.50$ whereas the spending multiplier is $\partial Y/\partial G = 1.5$.

e) For a given $\Delta G > 0$, find expressions for the effect of the expansionary fiscal policy on GBD_0 and B_1, respectively, in comparison with laissez-faire?

f) For a given r_1, and assuming that both $\partial Y/\partial G$ and $T'(Y)$ are approximately the same in period 1 as in period 0, find an expression for the effect of the expansionary fiscal policy on B_2 in comparison with laissez-faire?

Suppose the slump is over in period 2 and onwards whereby $G_t = \gamma Y_t$, $t = 2, 3, \ldots$. Suppose further that compared with the expansionary fiscal policy, laissez-faire during the slump would have implied not only higher unemployment, but also more people experiencing *long-term* unemployment. As a result some workers would have become de-qualified and in effect be driven out of the effective labor force. Suppose the loss in "full employment" output from period 2 and onwards implied by laissez-faire is ΔY per period, where ΔY is a positive constant.[2] Finally, let the ensuing loss in net tax revenue be $\tau \cdot \Delta Y$ per period, where τ is a positive constant (possibly close to $\hat{\tau}$ from b)).

g) With $r_t = r_2$, $t = 2, 3, \ldots$, and given ΔG and τ, find an expression for the value of ΔY required for the expansionary fiscal policy to "pay for itself" in period 2 and onwards in the sense that the averted loss in net tax revenue exactly offsets the extra interest payments?

h) Given $\tau = 0.29$, $r_1 = 0.01$, and $r_2 = 0.03$, answer again g). Comment.

B.25 Betragt en Diamond OLG model. Kald tidspræferenceraten (nyttediskonteringsraten) ρ og lad periodenyttefunktionen være specificeret som

$$u(c) = \frac{c^{1-\theta} - 1}{1 - \theta}, \qquad \theta > 0.$$

a) Kan tilfældet $\theta = 1$ fortolkes som meningsfuldt?

b) Udled opsparingsfunktionen for en ung tilhørende generation 0, når reallønnen i periode 0 er w_0 og realrenten (nettoafkastraten i periode 1 på den i periode 0 foretagne opsparing) er r_1.

c) Bestem forbruget som ung, c_0.

Betragt de to udsagn:

[2] It is theoretically possible that ΔY is more or less constant for a long time because two offsetting effects are operative. Because of technical progress the loss of output per lost worker is growing over time. On the other hand the pool of long-term unemployed generated by the slump will over time be a decreasing share of the labor force due to exit by the old and entrance by young people in the labor force.

(i) "En stigning i renteniveauet vil på c_0 have en negativ substitutionseffekt, en positiv (ren) indkomsteffekt og en negativ formueeffekt."

(ii) "Når $\theta > 1$, vil den totale effekt på c_0 af en stigning i renteniveauet være positiv."

d) Vurdér, om (i) er sand eller falsk, og om (ii) er sand eller falsk. Begrund din vurdering. *Vink:* Opskriv den intertemporale budgetrestriktion.

Vi skal nu sammenligne med Ramseymodellen. Med CRRA-nyttefunktion med parameter $\theta > 0$ fører Ramseymodellen til forbrugsfunktionen:

$$c_0 = \beta_0(a_0 + h_0),$$

hvor $a_0 > 0$ er en given finansiel formue per capita på tidspunkt 0, og

$$\begin{aligned}\beta_0 &\equiv \frac{1}{\int_0^\infty e^{\int_0^t (\frac{(1-\theta)r_\tau - \rho}{\theta} + n) d\tau} dt},\\ h_0 &\equiv \int_0^\infty w_t e^{-\int_0^t (r_\tau - n) d\tau} dt,\end{aligned}$$

hvor n er befolkningsvækstraten.

e) Betragt igen udsagnene (i) og (ii) og vurdér dem hver for sig ud fra Ramseymodellen. *Vink:* Opskriv den intertemporale budgetrestriktion.

B.26 Lad c_t være forbrugvareoutput pr. enhed arbejde i periode t, k_t kapital pr. enhed arbejde i periode t og k_{GR} golden rule-kapitalintensiteten. Betragt udsagnet: "I Diamonds OLG-model uden tekniske fremskridt må et teknisk muligt forløb, $\{k_t, c_t\}_{t=0}^\infty$, med egenskaben $k_t \to k^* < k_{GR}$, være dynamisk inefficient." Sandt eller falsk? Kommentér.

B.27 Betragt en økonomi beskrevet ved Diamond OLG model. Kald tidspræferenceraten (nyttediskonteringsraten) ρ og lad periodenyttefunktionen være specificeret som

$$u(c) = \frac{c^{1-\theta} - 1}{1-\theta}, \qquad \theta > 0, \theta \neq 1.$$

a) Udled opsparingsfunktionen for en ung tilhørende generation t, givet reallønnen w_t og realrenten r_{t+1}.

Lad L_t være antal unge tilhørende generation t og lad $n \geq 0$ være befolkningsvækstraten. Antag at produktionsfunktionen er Cobb-Douglas med konstant skalaafkast, og at der er Harrod-neutrale tekniske fremskridt med konstant rate $g > 0$.

b) Find udtryk for w_t og r_{t+1} i generel ligevægt ud fra henholdsvis $\tilde{k}_t$ og $\tilde{k}_{t+1}$, hvor $\tilde{k}$ er det teknologikorrigerede kapital-arbejdskraft-forhold.

c) Udled den fundamentale differensligning ("bevægelsesloven" for økonomien). *Vink:* $K_{t+1} = s_t L_t$.

d) Konstruér et transitionsdiagram til illustration af økonomiens udvikling over tid. *Vink:* I denne forbindelse er din generelle viden om CRRA-Cobb-Douglas-tilfældet nyttig.

e) Kan man for arbitrær initialværdi $\tilde{k}_0 > 0$ konkludere utvetydigt angående: (i) eksistens; (ii) entydighed; og (iii) stabilitet af en steady state? Begrund dine svar.

Antag økonomien har været i steady state til og med periode $t_0 - 1$, hvor $t_0 > 0$. Ved overgangen fra periode $t_0 - 1$ til periode t_0 sker der af en eller anden grund uventet et skift i de unges tidspræferencerate til et lavere niveau. Dette nye niveau forbliver gældende i alle perioder fra og med periode t_0.

f) Illustrér ved hjælp af det samme eller et nyt transitionsdiagram, hvordan transitions-kurven forskydes, og hvad økonomiens udvikling fra og med periode $t_0 - 1$ bliver. *Vink:* Du kan benytte din intuition, men det er bedst, hvis du også matematisk kan begrunde din konklusion; med henblik herpå er det nyttigt at omskrive steady state-ligningen, så $\tilde{k}$ kun optræder gennem potensfunktionen $\tilde{k}^{\alpha-1}$.

g) Lad y betegne output pr. enhed arbejde. I $(t, \ln y)$-planen tegn for $t \geq 0$ tidsprofilen for $\ln y$.

h) Hvordan, om overhovedet, påvirkes y's vækstrate af ændringen i tidspræferenceraten? Her kan det være en god idé at sondre imellem det relativt korte sigt og det lange sigt.

i) Forklar med ord den økonomiske mekanisme bag din konklusion ved h).

B.28 Vi ser på en lukket økonomi med offentlig sektor. Vi ignorerer konjunktursvingninger og antager, at BNI vokser eksogent med en given konstant rate $g_Y \geq 0$, og at realrenten er en given konstant $r > g_Y$. Tiden er diskret. Notationen er:

G_t er den reale offentlige udgift til varer og tjenester i periode t,
X_t er den reale offentlige udgift til indkomstoverførsler i periode t,
GBD_t er det reale offentlige budgetunderskud i periode t,
B_t er den reale offentlige gæld i starten af periode t.

Antag, at positive (negative) budgetunderskud altid udelukkende finansieres ved øget (mindsket) offentlig gæld, og at al indkomst beskattes med en given konstant skattesats $\tau \in (0, 1)$, hvilket giver brutto-skatteprovenuet

$$\tilde{T}_t = \tau(Y_t + X_t + rB_t). \qquad (*)$$

Vi benytter her symbolet $\tilde{T}_t$ for at undgå forveksling med nettoskatteprovenuet, $\tilde{T}_t - X_t$, der i forelæsningsnoterne som regel er betegnet T.

a) Stil to ligninger op, der viser, hvordan henholdsvis GBD_t og B_{t+1} er bestemt af variable dateret t.

Antag at $B_0 > 0$, $G_t = \gamma Y_t$, og $X_t = \chi Y_t$, $t = 0, 1, 2, \ldots$, hvor $\gamma \in (0, 1)$ er given, mens $\chi = (\tau - \gamma)/(1 - \tau)$, uanset hvad τ og γ er. Er $\tau < \gamma$, er $X_t < 0$, hvorved $-X_t$ kan fortolkes som provenuet af en yderligere skat (udover indkomstskatten, fx en lump-sum skat).

b) Bestem fortegnet på budgetunderskuddet i periode t for $t = 0, 1, 2, \ldots$.

c) Bestem udviklingen over tid i gældskvoten, b_t, defineret på den gængse måde som $b_t \equiv B_t/Y_t$. *Hint:* Udled en førsteordens differensligning for b_t og benyt herefter bagudrettet substitution.

d) Hvor stor vil τ skulle være for, at “eksploderende” b_t lige akkurat undgås?

Indtil videre lader vi kriteriet for finanspolitisk holdbarhed være, at b_t er “ikke-eksploderende”.

e) Antag, at $g_Y = 0$. Kan den førte finanspolitik, (τ, γ, χ), være holdbar? Hvorfor/hvorfor ikke?

f) Antag, at $g_Y > 0$. Kan den førte finanspolitik, (τ, γ, χ), være holdbar? Hvorfor/hvorfor ikke?

g) Antag nu, at renteindtægter ikke beskattes, så der i stedet for (*) gælder

$$\tilde{T}_t = \tau(Y_t + X_t),$$

mens vi stadig har $g_Y > 0$. Kan den førte finanspolitik, (τ, γ, χ), være holdbar? Hvorfor/hvorfor ikke? Kommentér.

Måske er den skattepligtige indkomst et mere relevant indkomstmål i denne sammenhæng end BNI. Vi lader derfor nu kriteriet for finanspolitisk holdbarhed være, at $\beta_t \equiv B_t/(Y_t + X_t + rB_t)$ er "ikke-eksploderende".

h) Besvar igen e). Kommentér.

Chapter C

Basic representative agent model in continuous time (the Ramsey model)

C.1 *The saving problem with a CRRA instantaneous utility function.* Consider a household with infinite horizon and a CRRA instantaneous utility function. Facing a constant real interest rate $r > 0$, the household solves the problem (standard notation):

$$
\begin{aligned}
\max_{(c_t)_{t=0}^{\infty}} U_0 &= \int_0^{\infty} \frac{c_t^{1-\theta}}{1-\theta} e^{-\rho t} dt \quad \text{s.t.} \\
c_t &\geq 0, \\
\dot{a}_t &= r a_t + w_t - c_t, \qquad a_0 \text{ given}, \\
\lim_{t\to\infty} a_t e^{-rt} &\geq 0,
\end{aligned}
$$

where $\theta > 0$.

a) Briefly, interpret the objective function and the constraints.

b) Derive the first-order conditions and the Keynes-Ramsey rule. Interpret.

c) Assuming a solution exists, solve the problem, i.e., find the consumption function. *Hint:* combine the Keynes-Ramsey rule with strict equality in the intertemporal budget constraint.

d) Your answer to c) was probably based on a set of necessary first-order conditions. Can you be sure they are also sufficient? Why or why not?

e) Does the CRRA utility function satisfy the No Fast Assumption? Why of why not?

C.2 *A positive technology shock.* A competitive economy is described by the two differential equations

$$\dot{\tilde{k}}_t = f(\tilde{k}_t) - \tilde{c}_t - (\delta + g + n)\tilde{k}_t, \qquad \tilde{k}_0 > 0 \text{ given}, \tag{*}$$

$$\dot{\tilde{c}}_t = \frac{1}{\theta}(f'(\tilde{k}_t) - \delta - \rho - \theta g)\tilde{c}_t, \tag{**}$$

and the condition

$$\lim_{t\to\infty} \tilde{k}_t e^{-\int_0^t (f'(\tilde{k}_s) - \delta - g - n)ds} = 0. \tag{***}$$

Notation is: $\tilde{k}_t \equiv K_t/(T_t L_t)$ and $\tilde{c}_t \equiv C_t/(T_t L_t) \equiv c_t/T_t$, where K_t and C_t are aggregate capital and aggregate consumption, respectively, and L_t is population = labor supply, all at time t. Further, T_t is a measure of the technology level and f is a CRS production function on intensive form, satisfying $f' > 0$, $f'' < 0$, and the Inada conditions. The remaining symbols stand for parameters and all these are strictly positive. Furthermore, $\rho - n > (1 - \theta)g$.

a) Briefly interpret the three above equations, including the parameters.

b) Draw a phase diagram and illustrate the path the economy follows, given some arbitrary positive initial value of $\tilde{k}$. Can the divergent paths be ruled out? Why or why not? Is dynamic inefficiency theoretically possible in this economy? Why or why not?

Assume the economy has been in steady state until time 0. Then an unanticipated technology shock occurs so that T_0 is replaced by $T_0' > T_0$. After this shock everybody rightly expects T to grow forever at the same rate as before. We now study short- and long-run effects of this shock.

c) Illustrate by means of the phase diagram what happens to $\tilde{k}$ and $\tilde{c}$ on impact, i.e., immediately after the shock, and in the long run.

d) What happens to the real interest rate on impact and in the long run?

e) Why is the sign of the impact effect on the real wage ambiguous (at the theoretical level) as long as f is not specified further?[1]

[1] Remark: for "empirically realistic" production functions (having elasticity of factor substitution larger than elasticity of production w.r.t. capital), the impact effect *is* positive, however (cf. Chapter 4 of Lecture Notes).

f) Compare the real wage in the long run to what it would have been without the shock.

g) Suppose $\theta = 1$. Why is the sign of the impact effect on per capita consumption ambiguous? *Hint:* $c = (\rho - n)(k + h)$.

h) Compare per capita consumption in the long run with what it would have been without the shock.

i) It is well-known that the equilibrium path generated by a Diamond OLG model can be dynamically inefficient. This is because the Diamond model violates one of the conditions needed for the First Welfare Theorem to hold. What is this condition and does the Ramsey model satisfy it? Why or why not?

C.3 *Aggregate saving, the return to saving.* Consider a Ramsey model, in continuous time, of a closed competitive market economy with public consumption, transfers, and capital income taxation. The government budget is always balanced. The model leads to the following differential equations (standard notation)

$$\begin{aligned}\dot{\tilde{k}}_t &= f(\tilde{k}_t) - \tilde{c}_t - \tilde{\gamma} - (\delta + g + n)\tilde{k}_t, \qquad \tilde{k}_0 > 0 \text{ given}, && (*)\\ \dot{\tilde{c}}_t &= \frac{1}{\theta}\Big[(1-\tau_r)(f'(\tilde{k}_t) - \delta) - \rho - \theta g\Big]\tilde{c}_t, && (**)\end{aligned}$$

and the condition

$$\lim_{t\to\infty} \tilde{k}_t e^{-\int_0^t \left[(1-\tau_r)(f'(\tilde{k}_s)-\delta)-g-n\right]ds} = 0. \qquad (***)$$

All parameters are positive and it is assumed that

$$\lim_{\tilde{k}\to 0} f'(\tilde{k}) - \delta > \frac{\rho + \theta g}{1-\tau_r} > n + g > \lim_{\tilde{k}\to\infty} f'(\tilde{k}) - \delta.$$

The government controls the technology-corrected per capita public consumption $\tilde{\gamma} \equiv G_t/(T_t L_t)$, the tax rate $\tau_r \in (0,1)$, and the transfers. Until further notice $\tilde{\gamma}$ and τ_r are kept constant over time and $\tilde{\gamma}$ is of "moderate" size so as to not rule out existence of a steady state. The transfers are continuously adjusted so that the government budget remains balanced.

a) Briefly interpret (*), (**), and (***), including the parameters.

b) Draw a phase diagram and illustrate the path that the economy follows, for a given $\tilde{k}_0 > 0$. Comment.

c) Is it possible for a steady state to exist without assuming f satisfies the Inada conditions? Why or why not?

d) Suppose the economy has been in steady state until time t_0. Then, suddenly τ_r is changed to a new constant level $\tau_r' < \tau_r$. The transfers are immediately adjusted so that the government budget remains balanced. Illustrate by a phase diagram what happens in the short and long run. Give an economic interpretation of your result.

e) Does the direction of movement of $\tilde{k}$ depend on θ? Comment.

f) Suppose $\theta = 1$. It is well-known that in this case the substitution effect and the income effect on current consumption of an increase in the (after-tax) rate of return offset each other. Can we from this conclude that aggregate saving does not change in response to the change in fiscal policy? Why or why not? *Hint:* when $\theta = 1$, $c_t = (\rho - n)(a_t + h_t)$, where

$$h_t \equiv \int_t^\infty (w_s + x_s) e^{-\int_t^s [(1-\tau_r)r_\tau - n]d\tau} ds;$$

here, x_s is per capita transfers at time s. Four partial equilibrium "effects" are in play, not only the substitution and income effects.

C.4 *Command optimum.* Consider a Ramsey setup with CRRA utility and exogenous technical progress at the constant rate $g \geq 0$. Suppose resource allocation is not governed by market mechanisms, but by a "social planner", an "all-knowing and all-powerful" central authority. The social planner is not constrained by other limitations than those coming from technical feasibility and can thus ultimately decide on the resource allocation within these confines.

The decision problem of the social planner is (standard notation):

$$\begin{aligned}
\max_{(c_t)_{t=0}^\infty} U_0 &= \int_0^\infty \frac{c_t^{1-\theta}}{1-\theta} e^{-(\rho-n)t} dt \qquad \text{s.t.} \\
c_t &\geq 0, \\
\dot{\tilde{k}}_t &= f(\tilde{k}_t) - \frac{c_t}{T_t} - (\delta + g + n)\tilde{k}_t, \qquad \tilde{k}_0 > 0, \text{ given}, \\
\tilde{k}_t &\geq 0 \qquad \text{for all } t \geq 0,
\end{aligned}$$

where $\delta + g > 0$, $\rho - n > (1-\theta)g$, and $\theta > 0$ (in case $\theta = 1$, the expression $c^{1-\theta}/(1-\theta)$ should be interpreted as $\ln c$). Assume that the production function satisfies the Inada conditions.

a) Briefly interpret the problem, including the parameters. Comment on the inequality $\rho - n > (1-\theta)g$.

b) Derive a characterization of the solution to the problem and illustrate by a phase diagram.

c) Compare the solution to the equilibrium path generated by a market economy described by a Ramsey model with perfect competition and with the same intertemporal utility function and the same technology as above. Comment.

d) Can the social planner's solution be dynamically inefficient? Why or why not?

e) Why does the social planner in the present problem not choose a path converging to the highest possible sustainable path for consumption per unit of labor?

f) Can the equilibrium path generated by a market economy as described under c) be dynamically inefficient? Why or why not?

g) Can the equilibrium path generated by a market economy as described by the Diamond OLG model be dynamically inefficient? Why or why not?

h) The true answer to f) and g) is not the same. Give an intuitive reason for this.

C.5 *Short questions.*

a) Suppose labor income is taxed at the rate $\tau_w \in (0, 1)$. Assume that the revenue from this income tax is redistributed back to the households as lump-sum transfers. Consider the assertion: "In the Ramsey model a time-varying labor income tax τ_w will distort the allocation of resources." True or false? Why?

In the next three questions we consider a Ramsey model with a consumption tax rate $\tau_c > 0$ such that the household pays $(1 + \tau_c)c$ for the consumption level c. The government revenue from the tax is used for financing lump-sum transfers to the households.

b) Let the level of τ_c be constant over time. Consider the assertion: "In the Ramsey model a constant consumption tax rate will distort the allocation of resources." True or false? Why?

c) Let the level of τ_c be constant over time. Suppose someone states the following: "In the described framework, an unanticipated once-for-all rise in the level of the consumption tax rate leads to lower consumption in the short run and higher capital intensity in the long run." True or false? Why?

d) Consider the assertion: "In the Ramsey model a time-varying consumption tax rate will distort the allocation of resources." True or false? Why?

C.6 *Short questions.* Consider the Ramsey model with CRRA utility and Harrod-neutral technical progress at rate g.

a) Can a path *below* the saddle path in $(\tilde{k}, \tilde{c})$ space be precluded as an equilibrium path with perfect foresight in the Ramsey model? Why or why not?

b) Can a path *above* the saddle path in $(\tilde{k}, \tilde{c})$ space be precluded as an equilibrium path with perfect foresight in the Ramsey model? Why or why not?

c) Answer questions a) and b) now presuming that "an equilibrium path with perfect foresight" is replaced by "a solution to the social planner's problem".

C.7 *Short questions.* Consider the optimization problem in C.1.

a) State by *words* what the No-Ponzi-Game condition says.

b) The No-Ponzi-Game condition belongs to problems with an infinite horizon. What is the analogue constraint for a problem with finite horizon?

c) State by *words* what the perfect foresight transversality condition of a household with infinite horizon says.

d) In a standard consumption/saving problem, is the household's transversality condition a constraint in the maximization problem or does it express a property of the solution?

C.8 *Short questions (functional income distribution, stylized facts, rate of return).*

a) "If and only if the production function is Cobb-Douglas, does the Ramsey model predict that the share of labor income in national income is constant in the long run." True or false? Give a reason for your answer.

b) Are predictions based on the Ramsey model (with exogenous Harrod-neutral technical progress) consistent with Kaldor's stylized facts? Why or why not?

c) Suppose we want a concise economic theory giving the long-run level of the average rate of return in the economy as an explicit or implicit function of only a few parameters and/or exogenous variables. Does the Ramsey model give us such a theory? Why or why not?

d) Briefly, assess the theory of the long-run rate of return implied by the Ramsey model. That is, mention what you regard as strengths and weaknesses of the theory.

C.9 *The saving problem with a CARA instantaneous utility function.* Consider a household with infinite horizon and a CARA instantaneous utility function. Facing a constant real interest rate $r > 0$, the household solves the problem (standard notation):

$$\begin{aligned} \max_{(c_t)_{t=0}^{\infty}} U_0 &= \int_0^{\infty} (-\beta^{-1} e^{-\beta c_t}) e^{-\rho t} dt \quad \text{s.t.} \\ c_t &\geq 0, \\ \dot{a}_t &= r a_t + w_t - c_t, \qquad \text{where } a_0 \text{ is given,} \\ \lim_{t\to\infty} a_t e^{-rt} &\geq 0, \end{aligned}$$

where $\beta > 0$.

a) Briefly, interpret the objective function and the constraints.

b) Derive the first-order conditions and find what form the Keynes-Ramsey rule takes.

c) Solve the problem, i.e., find the consumption function, presupposing that $r \geq \rho$ and initial total wealth, $a_0 + h_0$ (where h_0 is human wealth), is greater than $(r - \rho)/(\beta r^2)$. *Hint:* combine the Keynes-Ramsey rule with strict equality in the intertemporal budget constraint.

d) Your answer to c) was probably based on a set of necessary first-order conditions. Can you be sure they are also sufficient? Why or why not?

e) Does the CARA utility function satisfy the No Fast assumption? Why of why not?

C.10 This problem presupposes that you have solved Problem C.3. Consider the model in that problem. Suppose the economy has been in steady state until time t_0. Then the government credibly announces that at time $t_1 > t_0$, τ_r will be changed to a new constant level $\tau_r' < \tau_r$. Assume that people believe in this announcement and that the new taxation policy is implemented as announced. Illustrate by a phase diagram and graphic time profiles of $\tilde{c}$ and $\tilde{k}$ what happens in the economy for $t \geq t_0$. Explain.

C.11 *A budget deficit rule.* Let time be continuous and suppose that money financing of budget deficits never occurs. Consider a budget deficit rule saying that the *nominal* budget deficit must never be above $\alpha \cdot 100$ per cent of *nominal* GDP, PY, $\alpha > 0$, that is, the requirement is

$$\dot{D} \leq \alpha PY, \tag{*}$$

where $\dot{D} \equiv dD/dt$ (given $D = D(t)$ is nominal government debt) and $P = P(t)$ is a price index, whereas $Y = Y(t)$ is real GDP.

a) Is the deficit rule in the SGP of the EMU a special case of this? Why or why not?

b) Suppose the deficit rule (*) is always *binding* for the economy we look at. Derive the implied long-run value, b^*, of the debt-income ratio $b \equiv D/(PY)$, assuming a non-negative, constant inflation rate π (just a symbol for a constant, not necessarily the mathematical constant 3.14159...) and a positive constant growth rate, g_Y, of GDP. *Hint:* the differential equation $\dot{x} + ax = k$, where a and k are constants, $a \neq 0$, has the solution $x_t = (x_0 - x^*)e^{-at} + x^*$, where $x^* = k/a$.

c) Let the time unit be one year, $\pi = 0.02$, and $g_Y = 0.03$ for the SGP of the EMU. Calculate the value of b^*. Comment.

C.12 *Short questions*

a) What is meant by *Ricardian equivalence*?

b) What is the basic reason that the Diamond model and the Ramsey model give opposite conclusions as to the question of Ricardian equivalence?

C.13 *Short question*

"Considering the different Slutsky effects, the consumption function of the representative household in a Ramsey model with logarithmic instantaneous utility is such that a higher tax on interest income lowers current consumption." True or false? Why?

C.14 *Some quotations.*

a) Two economists – one from MIT and one from Chicago – are walking down the street. The MIT economist sees a 100 dollar note lying on the sidewalk and says: "Oh, look, what a fluke!". "Don't be silly, obviously it is false", laughs the Chicago economist, "if it wasn't, someone would have picked it up". Discuss in relation to the theoretical concepts of arbitrage and equilibrium.

b) A riddle asked by Paul Samuelson (Nobel Prize winner 1970): A physicist, a chemist, and an economist are stranded on an island, with nothing to eat. A can of soup washes ashore. The physicist says "let us smash the can open with a rock". The chemist says "let us build a fire and heat the can first". Guess what the economist says?

Chapter D

More applications of the Ramsey model

D.1 *Effects of a shift to higher patience in the population.* Consider a Ramsey model for a closed economy. The model can be reduced to two differential equations

$$
\begin{aligned}
\dot{\tilde{k}}_t &= f(\tilde{k}_t) - \tilde{c}_t - (\delta + g + n)\tilde{k}_t, \qquad \tilde{k}_0 > 0 \text{ given}, && (*) \\
\dot{\tilde{c}}_t &= \frac{1}{\theta}(f'(\tilde{k}_t) - \delta - \rho - \theta g)\tilde{c}_t, && (**)
\end{aligned}
$$

and the condition

$$\lim_{t\to\infty} \tilde{k}_t e^{-\int_0^t (f'(\tilde{k}_s) - \delta - g - n)ds} = 0. \qquad (***)$$

Notation is: $\tilde{k}_t = K_t/(T_t L_t)$ and $\tilde{c}_t = C_t/(T_t L_t) = c_t/T_t$, where K_t and C_t are aggregate capital and aggregate consumption, respectively, and L_t is population = labor supply, all at time t. Further, T_t is a measure of the technology level and f is a production function on intensive form, satisfying $f' > 0$, $f'' < 0$, and the Inada conditions. The remaining symbols stand for parameters and all these are positive. Moreover, $\rho - n > (1 - \theta)g$.

a) Briefly interpret the equations (*), (**), and (***), including the parameters.

b) Draw a phase diagram and illustrate the path the economy follows, given some arbitrary positive $\tilde{k}_0$. Can the divergent paths be ruled out? Why or why not?

c) Is dynamic inefficiency theoretically possible in this economy? Why or why not?

Assume the economy has been in steady state until time 0. Then for some external reason an unanticipated downward shift in the impatience parameter occurs. After this shock everybody rightly expects the impatience parameter to remain at its new lower level forever. We now study short- and long-run effects of this shock.

d) Illustrate by means of the phase diagram what happens to the economy on impact, i.e., immediately after the shock, and in the long run. Comment.

e) What happens to the real wage on impact and in the long run?

f) What happens to the real interest rate on impact and in the long run?

g) Does the long-run real interest rate depend on n? Why or why not?

h) If instead the economy is described by a Diamond OLG model with CRRA period utility and Cobb-Douglas production function, what will the answer to question g) be?

i) Compare your answer to question h) to the result in the Ramsey model. Comment.

D.2 Consider a Ramsey model, in continuous time, of a closed competitive market economy with public consumption, transfers, and a tax, τ, on financial *wealth* (not interest income). The wealth tax implies that the household's dynamic budget identity in per capita terms reads

$$\dot{a}_t = (r_t - \tau - n)a_t + w_t + x_t - c_t, \qquad a_0 \text{ given}, \tag{*}$$

where a_t is per capita financial wealth in the household, r_t the real interest rate, n the population growth rate, w_t the real wage, x_t a per capita lump-sum transfer, and c_t per capita consumption. The government budget is always balanced. The model leads to the following differential equations (standard notation)

$$\begin{aligned} \dot{\tilde{k}}_t &= f(\tilde{k}_t) - \tilde{c}_t - \tilde{\gamma} - (\delta + g + n)\tilde{k}_t, \qquad \tilde{k}_0 > 0 \text{ given}, && (**) \\ \dot{\tilde{c}}_t &= \frac{1}{\theta}(f'(\tilde{k}_t) - \delta - \tau - \rho - \theta g)\tilde{c}_t, && (***) \end{aligned}$$

and the condition

$$\lim_{t\to\infty} \tilde{k}_t e^{-\int_0^t (f'(\tilde{k}_s) - \delta - \tau - n - g)ds} = 0. \tag{****}$$

Assume that the production function satisfies the Inada conditions and that $\rho - n > (1-\theta)g$. With G_t denoting public consumption, T_t the technology level, and L_t the size of the labor force (= size of population), we assume (until further notice) that the government keeps $\tilde{\gamma} \equiv G_t/(T_t L_t) > 0$ as well as $\tau > 0$ constant over time and continuously adjusts the transfers so that the government budget remains balanced. Moreover, $\tilde{\gamma}$ is of "moderate" size relative to the production capacity of the economy so as to not rule out existence of a steady state. All parameters are nonnegative.

a) Briefly interpret (**), (***), and (****), including the parameters. What would the dynamic budget identity (*) look like if instead of the tax, τ, on financial wealth there were a tax, τ_r, on interest income (sometimes called capital income)?

b) Draw the phase diagram for the dynamic system (**)-(***) and illustrate, for a given $\tilde{k}_0 > 0$, the path followed by the economy. Comment. Can the divergent paths be ruled out as equilibrium paths? Why or why not?

c) Are the results consistent with Kaldor's stylized facts? Comment.

d) What is the welfare effect of the described tax-transfer policy compared with a policy where all government expenses are financed by lump-sum taxes? A brief answer which tells the sign of the welfare effect, based on a bit of verbal reasoning, is enough. *Hint:* because the model has a representative household, we can use the utility integral of the household as a welfare criterion.

Suppose the economy has been in steady state up until time $t_0 > 0$. Then, unexpectedly τ is changed to a new constant level τ', $0 < \tau' < \tau$, and transfers are adjusted accordingly. Let the government announce that the new tax-transfer scheme will be adhered to forever and let people rightly believe this.

e) Illustrate by a phase diagram (possibly a new one) what happens in the short and the long run, respectively. Give an economic interpretation of your result.

Assume instead that the change in the tax-transfer policy is introduced in the following way: it is announced (unexpectedly) at time t_0 that the new tax-transfer policy is to be implemented at time $t_1 > t_0$. People believe it and the new policy with $\tau' < \tau$ is implemented at time t_1 as announced.

f) Indicate in a phase diagram the evolution of the economy for $t \geq 0$. *Hint:* Consider the consumption plan of the forward-looking household as of time t_0. Think about where the economy must be as of time t_1. Be aware that in the time interval $[t_0, t_1)$, the "old dynamics" in a sense still hold.

g) By a time profile diagram showing the graphs $(t, \tilde{c}_t)$ and $(t, \tilde{k}_t)$, illustrate how $\tilde{c}$ and $\tilde{k}$ evolve over time. Interpret in words what happens.

D.3 Consider a Ramsey model for a closed economy with a public sector. Let the tax revenue come solely from labor income taxation. The tax revenue is used by the government to finance a non-rival public service, G_t, as well as a lump-sum per capita transfer, x_t.

The representative household faces the problem (standard notation):

$$\begin{aligned}
\max_{(c_t)_{t=0}^{\infty}} U_0 &= \int_0^{\infty} (\frac{c_t^{1-\theta}}{1-\theta} + v(G_t))e^{-(\rho-n)t}dt \text{ s.t.} \\
c_t &\geq 0, \\
\dot{a}_t &= (r_t - n)a_t + (1-\tau)w_t + x_t - c_t, \qquad a_0 \text{ given}, \\
\lim_{t\to\infty} a_t e^{-\int_0^t (r_s - n)ds} &\geq 0,
\end{aligned}$$

where $\theta > 0$, $v' > 0$, and τ is a given constant labor income tax rate satisfying $0 < \tau < 1$.

a) Briefly, interpret the objective function and the constraints.

b) Derive the first-order conditions and the implied Keynes-Ramsey rule.

GDP is produced by an aggregate neoclassical CRS production function,

$$Y_t = F(K_t, T_t L_t),$$

where K_t and L_t are input of capital and labor, respectively, and T_t is the exogenous technology level, assumed to grow at the constant rate $g > 0$. We assume that

$$\rho - n > (1-\theta)g.$$

We further assume that G_t is proportional to the work force measured in efficiency units, that is $G_t = \tilde{\gamma} T_t L_t$, where $\tilde{\gamma} > 0$ is decided by the government. The transfers are endogenous in the sense of being continuously adjusted so that the government budget remains balanced, i.e.,

$$G_t + x_t L_t = \tau w_t L_t \quad \text{for all } t.$$

The model can now be reduced to the following dynamic system

$$\dot{\tilde{k}}_t = f(\tilde{k}_t) - \tilde{c}_t - \tilde{\gamma} - (\delta + g + n)\tilde{k}_t, \qquad \tilde{k}_0 > 0 \text{ given}, \tag{*}$$

$$\dot{\tilde{c}}_t = \frac{1}{\theta}\left[f'(\tilde{k}_t) - \delta - \rho - \theta g\right]\tilde{c}_t, \tag{**}$$

where $\tilde{k}_t \equiv K_t/(T_t L_t) \equiv k_t/T_t$, $\tilde{c}_t \equiv c_t/T_t$, and f is the production function on intensive form, and where the condition

$$\lim_{t\to\infty} \tilde{k}_t e^{-\int_0^t \left(f'(\tilde{k}_s) - \delta - n - g\right)ds} = 0 \tag{***}$$

must be satisfied.

c) Briefly interpret (*), (**), and (***).

From now we assume that F satisfies the Inada conditions and that $\tilde{\gamma}$ is of "moderate" size so as to not rule out existence of a steady state.

d) Draw a phase diagram and illustrate the path the economy follows, given an arbitrary $\tilde{k}_0 > 0$. Can the divergent paths be ruled out? Why or why not? Is dynamic inefficiency theoretically possible in this economy? Why or why not?

Suppose the economy has been in steady state until time $t_0 > 0$. Then, unexpectedly, a new spending policy $\tilde{\gamma}' > \tilde{\gamma}$ is introduced without any change in τ. We assume that the households rightly expect this new policy, $(\tilde{\gamma}', \tau)$, to be maintained for all $t \geq t_0$.

e) Illustrate by the phase diagram (possibly a new one) what happens as a result of the policy shift.

f) Draw the time profile of $\tilde{k}_t$ and $\tilde{c}_t$ for $t \geq 0$ (the graphs $(t, \tilde{k}_t)$ and $(t, \tilde{c}_t)$). Give an economic interpretation of your result.

Now suppose instead that the rise in G at time t_0 is immediately accompanied by a change in the labor income tax rate to a level $\tau' > \tau$ so that the per capita transfer x need not be changed, at least not at time t_0. We assume that the households rightly expect this policy, $(\tilde{\gamma}', \tau')$, to be maintained for all $t \geq t_0$.

g) Illustrate by the phase diagram (possibly a new one) and by time profiles of $\tilde{k}_t$ and $\tilde{c}_t$ for $t \geq 0$ what happens as a result of this policy shift. Comment.

We now consider still another scenario. At time t_0, it is unexpectedly announced that the new policy, $(\tilde{\gamma}', \tau')$, will be implemented at time $t_1 > t_0$. Suppose people believe in this announcement and that when time t_1 arrives, the new policy is implemented as announced.

h) Illustrate by the phase diagram (possibly a new one) and by time profiles of $\tilde{k}_t$ and $\tilde{c}_t$ for $t \geq 0$ what happens as a result of this policy shift. Comment.

D.4 Betragt en Ramsey model for en lukket økonomi med offentlig sektor. Tiden er kontinuert. Der er et offentligt forbrug, G_t, lump-sum indkomstoverførsler, X_t, samt kapitalindkomstbeskatning med skattesatsen τ. Modellen leder frem til følgende to differentialligninger (standard notation):

$$\begin{aligned} \dot{\tilde{k}}_t &= f(\tilde{k}_t) - \tilde{c}_t - \tilde{\gamma} - (\delta + g + n)\tilde{k}_t, \qquad \tilde{k}_0 > 0 \text{ given,} && (*) \\ \dot{\tilde{c}}_t &= \frac{1}{\theta}\left[(1-\tau)(f'(\tilde{k}_t) - \delta) - \rho - \theta g\right]\tilde{c}_t, && (**) \end{aligned}$$

samt betingelsen

$$\lim_{t\to\infty} \tilde{k}_t e^{-\int_0^t \left[(1-\tau)(f'(\tilde{k}_s)-\delta)-g-n\right]ds} = 0. \qquad (***)$$

Alle parametre er positive, og det antages, at

$$\lim_{\tilde{k}\to 0} f'(\tilde{k}) - \delta > \frac{\rho + \theta g}{1-\tau} > n + g > \lim_{\tilde{k}\to\infty} f'(\tilde{k}) - \delta.$$

Regeringen fastsætter skattesatsen $\tau \in (0,1)$ og det teknologi-korrigerede offentlige per capita forbrug $\tilde{\gamma} \equiv G_t/(T_t L_t)$. Indtil videre er τ og $\tilde{\gamma}$ konstante; $\tilde{\gamma}$ er af "moderat" størrelse således, at en steady steady kan eksistere. Lump-sum per capita indkomstoverførslerne, $x_t = X_t/L_t$, tilpasses kontinuerligt således, at det offentlige budget forbliver balanceret.

a) Giv en kortfattet fortolkning af (*), (**) og (***), inklusiv parametrene.

b) Konstruér et fasediagram og illustrér den bane, økonomien vil følge for given $\tilde{k}_0 > 0$. Kommentér.

c) Behøver vi at forudsætte Inadabetingelserne for at sikre eksistens af en steady state. Hvorfor/hvorfor ikke?

Antag, at økonomien har været i steady state indtil tidspunkt $t_0 > 0$. På tidspunkt t_0 ændres uventet τ til et lavere konstant niveau $\tau' \in (0, \tau)$. Indkomstoverførslerne tilpasses med det samme således, at det offentlige budget forbliver balanceret.

d) Husholdningerne forventer (med rette antager vi), at den nye skattesats vil blive fastholdt for alle $t \geq t_0$. Illustrér med et fasediagram (eventuelt et nyt), hvad der sker i tidsintervallet $[t_0, \infty)$. Giv en økonomisk fortolkning.

e) Antag i stedet, at regeringen troværdigt på tidspunkt t_0 meddeler, at skattesænkningen kun vil vare frem til tidspunkt $t_1 > t_0$, hvorefter skattesatsen vil være tilbage på sit gamle niveau i al fremtid. Under forudsætning af at husholdningerne tror herpå, illustrér med et fasediagram (eventuelt et nyt), hvad der sker i tidsintervallet $[t_0, \infty)$.

f) På basis af et diagram med graferne $(t, \tilde{c}_t)$ og $(t, \tilde{k}_t)$ illustrér udviklingen i $\tilde{c}$ og $\tilde{k}$ i tidsintervallet $[0, \infty)$ og giv en økonomisk fortolkning.

D.5 Antag, at hvert land i en gruppe af lande (for enkelheds skyld betragtet som lukkede økonomier) nogenlunde dækkende kan beskrives med Ramseymodellen med CRRA-nyttefunktion og Harrod-neutrale tekniske fremskridt. Antag videre, at landene har samme produktionsfunktion, F, samme initiale teknologiniveau, T_0, samme teknologivækstrate, $g > 0$, samme kapitalnedslidningsrate, δ, samme tidspræferencerate, ρ, samme befolkningsvækstrate, n, men forskellig grænsenytteelasticitet m.h.t. forbrug, θ. For hvert land er samtlige parameterværdier sådan, at et ligevægtsforløb (equilibrium path) eksisterer. Betragt udsagnet: "Modellen forudsiger på dette grundlag, at forskelle i landenes indkomst pr. indbygger kun vil være midlertidige." Sandt eller falsk? Kommentér.

D.6 *Korte spørgsmål*

a) Hvad forstås ved Ricardiansk ækvivalens?

b) Har Diamonds OLG-model egenskaben Ricardiansk ækvivalens? Hvad er den teoretiske baggrund for, at modellen har/ikke har denne egenskab?

c) Har Ramseymodellen egenskaben Ricardiansk ækvivalens? Hvad er den teoretiske baggrund for, at modellen har/ikke har denne egenskab?

Chapter E

A carbon tax. The q-theory of investment

E.1 *A carbon tax and Tobin's q* We consider a small open economy (henceforth called SOE) with perfect mobility of financial capital but no mobility of labor. The SOE faces a constant real interest rate $r > 0$, given from the world market for financial capital. The technology of the representative firm is given by a neoclassical production function with constant returns to scale,

$$Y_t = F(K_t, L_t, M_t), \quad F_i > 0, F_{ii} < 0 \quad \text{for} \quad i = K, L, M.$$

Here Y is output gross of installation costs, imports, and physical capital depreciation, K is capital input, and L is labor input, whereas M is an imported fossil energy source, say oil. We also assume that the three inputs are *direct complements* in the sense that

$$F_{ij} > 0, \quad i \neq j.$$

The firm faces strictly convex capital installation costs and the installation cost function is homogeneous of degree one: $J_t = G(I_t, K_t) \equiv K_t g(I_t/K_t)$, $g(0) = g'(0) = 0$, $g'' > 0$.

In national accounting what is called Gross Domestic Output (GDP) is aggregate gross value added, i.e.,

$$GDP_t = Y_t - J_t - p_M M_t, \tag{1}$$

where $p_M > 0$ is the exogenous real price of oil which we treat as a shift parameter.

The labor force of the SOE is a constant $\bar{L}$. There is perfect competition in all markets. There is a tax, $\tau > 0$, on use of fossil energy (a "carbon tax").

In equilibrium with full employment the following holds:

$$M_t = M(K_t, (1+\tau)p_M), \qquad M_K > 0, M_{p_M} < 0. \tag{2}$$

We write the marginal product of capital ("Marginal Product of K") as a function $MPK(K_t, (1+\tau)p_M) \equiv F_K(K_t, \bar{L}, M(K_t, (1+\tau)p_M))$. It can be shown that

$$MPK_K < 0, MPK_{p_M} < 0. \tag{3}$$

a) Set up the value maximization problem and derive the first-order conditions.

b) Given the general information put up, briefly explain by words why GDP takes the form in (1), why the partial derivatives in (2) *must* have the shown signs, why we *must* have $MPK_{p_M} < 0$ as indicated in (3), and why, at first glance, the sign of MPK_K might seem ambiguous.[1]

The dynamics of the capital stock is given by

$$\dot{K}_t = (m(q_t) - \delta)K_t, \qquad K_0 > 0 \text{ given}, \tag{4}$$

where $m(1) = 0$, $m' = 1/g''$, and δ is the capital depreciation rate whereas q is the shadow price of installed capital along the optimal path, satisfying the differential equation

$$\dot{q}_t = (r+\delta)q_t - MPK(K_t, (1+\tau)p_M) + g(m(q_t)) - m(q_t)(q_t - 1). \tag{5}$$

Moreover, a necessary transversality condition is

$$\lim_{t\to\infty} K_t q_t e^{-rt} = 0.$$

c) Briefly interpret (4) and (5): what is the economic "story" behind these equations?

d) Assuming F satisfies the Inada conditions, construct a phase diagram for the system (4) - (5). *Hint:* it can be shown that at least in a neighborhood of the steady state, the slope of the $\dot{q} = 0$ locus is negative.[2]

e) For an arbitrary $K_0 > 0$, indicate in the diagram the evolution of the pair (K_t, q_t) in general equilibrium. Does the convergent solution path satisfy the transversality condition? Is the solution to the model unique? *Hint:* it can be shown that the divergent solution paths violate the transversality condition.

[1] You do not have to explain why nevertheless $MPK_K < 0$ since it takes some steps to prove this. A proof is given in the appendix to Chapter 15 of the lecture notes.

[2] Indeed, the proof in Appendix E to Chapter 14 of the lecture notes is also valid here.

f) Assume that until time $t_0 > 0$, the economy has been in its steady state. Then, unexpectedly the government raises the carbon tax to $\tau' > \tau$. Illustrate graphically what happens on impact and gradually over time. Comment on the effect of the tax rise on investment.

g) Illustrate in another figure the time profiles of K_t and q_t for $t \geq 0$. Briefly explain in words.

h) We now go a little outside the present simple model. Suppose, that a non-fossil energy source is available whose application requires a lot of specially designed capital equipment. Although before the tax rise this alternative technology has not been privately cost-efficient, after the tax rise it is. Moreover, the representative firm expects internal positive learning-by-doing effects by adopting the alternative technology. Do you think it is possible that the rise in the carbon tax could in this broader setup affect capital investment in an opposite direction to what you found in e)? Why or why not?

E.2 In this exercise we consider an economy as described in Exercise E.1, ignoring its last question. Assume

$$G(I, K) = \frac{1}{2}\beta\frac{I^2}{K}, \qquad \beta > 0.$$

Assume further that until time $t_0 > 0$, the economy has been in its steady state with carbon tax equal to τ. Then, unexpectedly the government raises the carbon tax to $\tau' > \tau$ at the same time as it introduces an investment subsidy σ, $0 < \sigma < 1$, so that to attain an investment level I, purchasing the investment goods involves a cost of $(1-\sigma)I$. The subsidy is financed by some tax not affecting firms' behavior (for example a constant tax on households' consumption).

a) Given τ', find the required σ in terms of $MPK(K^*, (1 + \tau')p_M)$ such that the economy stays with its "old" steady-state capital stock, K^*. Comment.

b) How, if at all, is the steady-state value of q affected by the change in fiscal policy?

E.3 Consider a single firm with production function

$$Y_t = F(K_t, L_t),$$

where Y_t, K_t, and L_t are output, capital input, and labor input per time unit at time t, respectively, while F is a neoclassical production function with CRS and satisfying the Inada conditions. Time is continuous. The increase per time unit in the firm's capital stock is given by

$$\dot{K}_t = I_t - \delta K_t, \qquad \delta > 0,$$

where I_t is gross investment per time unit at time t and δ is the capital depreciation rate. There is perfect competition in all markets and there is no uncertainty. The real interest rate faced by the firm is a positive constant r. Cash flow (in real terms) at time t is

$$R_t = F(K_t, L_t) - w_t L_t - I_t - G(I_t),$$

where w_t is the real wage and $G(I_t)$ is a capital installation cost function satisfying

$$G(0) = G'(0) = 0, \; G''(I) > 0.$$

(An example is $G(I) = (\beta/2)I^2$, $\beta > 0$.)

a) Set up the firm's intertemporal production and investment problem as a standard optimal control problem, given that the firm wants to maximize its market value. Let the adjoint variable be denoted q_t.

b) Derive the first-order conditions and state the necessary transversality condition, TVC. *Hint:* the TVC has the standard form for an infinite horizon optimal control problem with discounting.

c) What is the economic interpretation of q_t?

d) Show that the optimal labor input is such that the capital-labor ratio at time t is an increasing function of w_t. *Hint:* it is convenient to consider the production function on intensive form.

Suppose from now that $w_t = w$ for all $t \geq 0$, where w is a positive constant. Let the corresponding optimal capital-labor ratio be denoted $\bar{k}$.

e) The optimal investment level, I_t, can be written as an implicit function of q. Show this. Construct a phase diagram for the (K, q) dynamics, assuming that a steady state with $K > 0$ exists. Let the steady state value of K be denoted K^*. *Hint:* since the capital installation cost function is simpler than usual, the phase diagram may look somewhat different from the usual one.

f) For an arbitrary $K_0 > 0$, indicate in the diagram the movement of the pair (K_t, q_t) along the optimal path. In another diagram draw the time profiles of q_t, I_t, and K_t. Comment on why, in spite of the marginal productivity of capital in the steady state exceeding $r + \delta$, there is no incentive to increase K above K^*.

g) It is common to call K^* the "desired capital stock". Express the desired capital stock as an implicit function of r, δ, and $\bar{k}$. How does the desired capital stock depend on r and w, respectively? Indicating the sign is enough. *Hint:* a simple approach can be based on curve shifting.

h) Show that optimal net investment, $I_t^n \equiv I_t - \delta K_t$, equals $\delta(K^* - K_t)$. In the obtained net investment rule you should recognize a principle from introductory macroeconomics. What is the name of this principle? Comment.

i) Let F be Cobb-Douglas with CRS and let $G(I) = (\beta/2)I^2$, $\beta > 0$. Find q, I, and K along the optimal path. *Hint:* the differential equation $\dot{x}(t) + ax(t) = b$ with $a \neq 0$ has the solution $x(t) = (x(0) - x^*)e^{-at} + x^*$, where $x^* = b/a$.

E.4 *Tobin's q and imperfect competition.* Consider a single firm supplying a differentiated good in the amount y_t per time unit at time t. The production function is

$$y_t = K_t^{\alpha} L_t^{1-\alpha}, \qquad 0 < \alpha < 1, \tag{*}$$

where K_t and L_t are capital and labor input at time t, respectively.

The nominal wage and the nominal *general* price level in the economy faced by the firm are constant over time and exogenous to the firm. So the real wage is an exogenous positive constant, w. The demand, y^d, for the firm's output is perceived by the firm as given by

$$y^d = p^{-\varepsilon}\frac{Y}{n}, \qquad \varepsilon > 1, \tag{**}$$

where p is the price set in advance by the firm (as a markup on expected marginal cost), relative to the general price level in the economy, n is the given large number of monopolistically competitive firms in the economy, Y is the overall level of demand, and ε is the (absolute) price elasticity of demand. The interpretation is that the firm faces a downward sloping demand curve the position of which is given by the general level of demand, which is exogenous to the firm. We assume that within the time horizon relevant for the analysis, Y is constant and the firm keeps p fixed, possibly due to menu costs. Moreover, the analysis will ignore uncertainty.

The increase per time unit in the firm's capital stock is given by

$$\dot{K}_t = I_t - \delta K_t, \qquad \delta > 0, \qquad K_0 > 0 \text{ given,}$$

where I_t is gross investment per time unit at time t and δ is the capital depreciation rate. We assume that p is high enough to always be above actual marginal cost so that it always pays the firm to satisfy demand. Then cash flow at time t is

$$R_t = py^d - wL_t - I_t - G(I_t),$$

where $G(I_t)$ is a capital installation cost function satisfying

$$G(0) = G'(0) = 0, \; G''(I) > 0.$$

a) To obtain $y_t = y^d$, a certain employment level is needed. Find this employment level as a function of K_t and y^d. Let your result be denoted $L(K_t, y^d)$.

The real interest rate faced by the firm is denoted r and is, until further notice, a positive constant. As seen from time 0, the firm solves the following decision problem:

$$\max_{(I_t)_{t=0}^{\infty}} V_0 = \int_0^{\infty} \left[py^d - wL(K_t, y^d) - I_t - G(I_t)\right] e^{-rt} dt \quad \text{s.t.}$$

$$I_t \text{ free (i.e., no restriction on } I_t),$$

$$\dot{K}_t = I_t - \delta K_t, \qquad K_0 > 0 \text{ given,}$$

$$K_t \geq 0 \text{ for all } t.$$

b) Briefly interpret this decision problem.

c) Denoting the adjoint variable q, derive the first-order conditions and state the necessary transversality condition (TVC) for a solution. *Hint:* the TVC has the standard form for an infinite horizon optimal control problem with discounting.

d) The optimal investment level, I_t, can be written as an implicit function of q_t. Show this.

e) Construct a phase diagram for the (K, q) dynamics, assuming that a steady state with $K > 0$ exists. Let the steady state value of K be denoted K^*. For an arbitrary $K_0 > 0$, indicate in the diagram the movement of the pair (K_t, q_t) along the optimal path.

Assume that until time t_1, the economy has been in steady state. Then, unexpectedly, the aggregate demand level, and thereby y^d, shifts to a new constant level $y^{d\prime} < y^d$ and is rightly expected to remain at that level for a long time.

f) Illustrate by the same or a new phase diagram what happens on impact and gradually over time. Comment on the implied effect on investment on impact and in the long run.

Assume instead that it is the interest rate which at time t_1 shifts to a new constant level $r' > r$ and is rightly expected to remain at that level for a long time.

g) Illustrate by the same or a new phase diagram what happens on impact and gradually over time. Comment on the implied effect on investment on impact and in the long run.

h) As a modified scenario, imagine that the fall in demand at time t_1 considered under f) was in fact due to a rise in the interest rate at time t_1. Compare the implied combined effect on investment on impact and in the long run with the isolated effects under f) and g), respectively.

i) Relate the results in f) and g) to the signs of the partial derivatives of the investment function in a standard IS-LM model. Comment.

E.5 Consider a single firm with production function

$$Y_t = K_t^{\alpha}(T_t L_t)^{1-\alpha}, \qquad 0 < \alpha < 1,$$

where Y_t, K_t, and L_t are output, capital input, and labor input per time unit at time t, respectively. Time is continuous and T_t is the technology level, growing over time at a constant rate $\gamma > 0$. The increase per time unit in the firm's capital stock is given by

$$\dot{K}_t = I_t - \delta K_t, \qquad K_0 > 0, \qquad \delta > 0,$$

where I_t is gross investment per time unit at time t and δ is the capital depreciation rate. Cash flow (in real terms) at time t is

$$R_t = K_t^{\alpha}(T_t L_t)^{1-\alpha} - J_t - w_t L_t - I_t,$$

where w_t is the real wage and J_t represents capital installation costs given by

$$J_t = \beta \frac{I_t^2}{2K_t}.$$

There is perfect competition in all markets and no uncertainty. The real interest rate faced by the firm is a constant $r > 0$.

a) Set up the firm's intertemporal production and investment problem as a standard optimal control problem, given that the firm wants to maximize its market value.

Let the adjoint variable be denoted q_t.

b) Derive the first-order conditions and state the necessary transversality condition (TVC). *Hint:* along the optimal plan the partial derivative of the current-value Hamiltonian w.r.t. the state variable equals the difference between the discount rate multiplied by the adjoint variable and the time derivative of the adjoint variable; the TVC has the standard form for an infinite horizon optimal control problem with discounting.

c) What is the economic interpretation of q_t? On the basis of one of the first-order conditions, express the optimal investment level at time t as a function of q_t and K_t.

From now on, assume that the firm is a representative firm in a small open economy.

d) Suppose the government wants to stimulate firms' investment and from time t_0 on implements a subsidy σ, $0 < \sigma < 1$, so that to attain an investment level I, purchasing the investment goods involves a cost of $(1 - \sigma)I$. Assuming the subsidy is financed by some tax not affecting firms' behavior (for example a tax on households' consumption), will the government attain its goal? Make sure that you substantiate your answer by a formal proof.

Define $\tilde{k}_t \equiv K_t/(T_t L_t)$ and assume that labor supply grows at the constant rate $n \geq 0$.

e) Show that $\dot{\tilde{k}}_t = (aq_t - b)\tilde{k}_t$, where a and b are constants.

f) On the basis of another of the first-order conditions from question b), derive an equation for $\dot{q}_t$ in terms of q_t and $\tilde{k}_t$.

g) Suppose $r > \gamma + n$. Draw a phase diagram in the $(\tilde{k}, q)$ plane and illustrate the evolution of the economy for $t \geq 0$, assuming that $0 < \tilde{k}_0 < \tilde{k}^*$, where $\tilde{k}^*$ is the steady-state value of $\tilde{k}$. *Hint:* it can be shown that in a neighborhood of the steady state, the $\dot{q} = 0$ locus is negatively sloped.

E.6 Consider a small open economy facing a constant real interest rate, given from the world market. Markets are competitive. Labor supply is inelastic and constant over time and there is no technical progress. The government contemplates introduction of an 'investment subsidy', σ, such that to buy I machines, each with a price equal to one unit of account, firms have to pay $(1-\sigma)I$ units of account, where σ is a constant, $0 < \sigma < 1$. The private sector is in a steady state and is not aware of these governmental considerations. "In this setting, Tobin's q-theory of investment predicts that by introducing and maintaining the investment subsidy σ, the government will be able to stimulate aggregate net investment temporarily, but not permanently." True or false? Why?

E.7 *Short question.*

In a standard Ramsey model and a model based on the q-theory of investment the circumstances under which firms optimize are different. Give a brief characterization of this difference and its implications.

E.8 *Short question.*

In many simple macroeconomic models a firm's acquisition of its capital input is described as if the firm solves a sequence of static profit maximization problems. One can imagine circumstances where this description of firms' behavior is not adequate, however. Give a brief account of what such circumstances might be and what alternative approach might be relevant.

E.9 Betragt en enkelt virksomhed, der står over for beslutningsproblemet: vælg en plan $(L_t, I_t)_{t=0}^{\infty}$, der maksimerer

$$V_0 = \int_0^\infty \left(F(K_t, L_t) - G(I_t, K_t) - w_t L_t - (1-\sigma)I_t\right) e^{-rt} dt \quad \text{u.b.} \tag{1}$$

$$L_t \geq 0, I_t \text{ fri (dvs. ingen restriktion på } I_t), \tag{2}$$

$$\dot{K}_t = I_t - \delta K_t, \qquad K_0 > 0 \text{ given}, \tag{3}$$
$$K_t \geq 0 \text{ for alle } t. \tag{4}$$

Her er F en neoklassisk produktionsfunktion med konstant skalaafkast og opfyldende Inada-betingelserne. Input af kapital og arbejde betegnes henholdsvis K_t og L_t; G er en funktion, der bestemmer kapitalinstalleringsomkostningerne, I_t er bruttoinvesteringer, w_t er en given realløn, $r > 0$ er en given konstant realrente, og $\delta > 0$ er en given konstant kapitalnedslidningsrate. Der er et konstant investeringssubsidium $\sigma \in (0, 1)$. Installeringsomkostningsfunktionen G opfylder

$$G(0, K) = 0,\ G_I(0, K) = 0,\ G_{II}(I, K) > 0, \quad \text{og} \quad G_K(I, K) \leq 0,$$

for alle (I, K).

a) Opstil løbende-værdi-Hamiltonfunktionen. Lad den til K knyttede hjælpevariabel (adjoint variable) være betegnet q_t. Udled førsteordensbetingelserne og angiv den nødvendige transversalitetsbetingelse. *Vink:* I dette problem har den nødvendige transversalitetsbetingelse standardformen ved et optimal-kontrol-problem med diskontering og uendelig tidshorisont.

b) Fortolk q_t. Vis at det optimale investeringsomfang til tidspunkt t er en funktion af q_t, K_t og σ.

Fra nu af ser vi på tilfældet $G(I, K) = \beta \frac{I^2}{2K}$, hvor $\beta > 0$.

c) Udtryk det optimale I_t/K_t som en funktion af q_t og σ.

Antag at den betragtede virksomhed er den repræsentative virksomhed i en lille åben økonomi med fuldkommen konkurrence og fri bevægelighed af finansiel kapital, men ingen bevægelighed af arbejdskraft på tværs af grænserne. Arbejdsstyrken i den lille åbne økonomi er en konstant, $\bar{L}$.

d) Vis at virksomhedens førsteordensbetingelser kombineret med fuld betigelse resulterer i to sammenhørende differentialligninger i K_t og q_t.

e) Konstruér det tilhørende fasediagram og vis i diagrammet (K_t, q_t)'s udvikling over tid for arbitrær initialværdi $K_0 > 0$. Kommentér.

f) Antag at indtil tidspunkt $t_0 > 0$ har systemet været i steady state med $(K, q) = (K^*, q^*)$. Så hæver regeringen uventet investeringssubsidiet til niveauet $\sigma' > \sigma$. Antag at investeringssubsidiet med rette forventes at forblive på det nye niveau i al fremtid. Illustrér ved hjælp af det samme eller et nyt fasediagram den bane, som (K_t, q_t) følger for $t > t_0$. Kommentér.

g) Betragt udsagnet: “Stigningen i investeringssubsidiet har en midlertidig virkning på bruttoinvesteringerne, ikke en permanent.” Sandt eller falsk? Hvorfor?

Chapter F

Uncertainty, expectations, and speculative bubbles

F.1 Suppose that $Y_t = X_t + e_t$, where $\{X_t\}$ is a random walk and e_t is white noise.

a) Calculate the rational expectation of X_t and Y_t conditional on all relevant information up to and including period $t-1$?

b) Compare the rational expectation of Y_t with the subjective expectation of Y_t based on the adaptive expectations formula with adjustment speed equal to one.

c) Compare the rational expectation of X_t with the subjective expectation of X_t based on the adaptive expectations formula with adjustment speed equal to one.

F.2 Consider a simple Keynesian model of a closed economy with constant wages and prices (behind the scene), abundant capacity, and output determined by demand:

$$
\begin{aligned}
Y_t &= D_t = C_t + \bar{I} + G_t, && && (1)\\
C_t &= \alpha + \beta Y^e_{t-1,t}, && \alpha > 0,\ 0 < \beta < 1, && (2)\\
G_t &= (1-\rho)\bar{G} + \rho G_{t-1} + \varepsilon_t, && \bar{G} > 0,\ 0 < \rho < 1, && (3)
\end{aligned}
$$

where the endogenous variables are Y_t = output (= income), D_t = aggregate demand, C_t = consumption, and $Y^e_{t-1,t}$ = expected output (income) in period t as seen from period $t-1$, while G_t, which stands for government spending on goods and services, is considered exogenous as is ε_t, which is white noise.

Finally, investment, $\bar{I}$, and the parameters α, β, ρ, and $\bar{G}$ are given positive constants.

Suppose expectations are "static" in the sense that expected income in period t equals actual income in the previous period.

a) Solve for Y_t.

b) Find the income multiplier (partial derivative of Y_t) with respect to a change in G_{t-1} and ε_t, respectively.

Suppose instead that expectations are rational.

c) Explain what this means.

d) Solve for Y_t.

e) Find the income multiplier with respect to a change in G_{t-1} and ε_t, respectively.

f) Compare the result under e) with that under b). Comment.

Consider a simple Keynesian model of a closed economy with constant wages and prices (behind the scene), abundant capacity, and output determined by demand:

$$\begin{aligned} Y_t &= C_t + I_t, && (1)\\ C_t &= a + bY^e_{t-1,t}, \qquad a > 0, 0 < b < 1, && (2)\\ I_t &= \bar{I}_t + \varepsilon_t, \qquad \bar{I}_t > 0, && (3) \end{aligned}$$

where the endogenous variables are Y_t = output (= income), C_t = consumption, I_t = investment, and $Y^e_{t-1,t}$ = expected output in period t as seen from period $t-1$. The variable ε_t is white noise. To simplify, $\bar{I}_t$ is exogenous (predetermined) and known by the public in advance.

Suppose expectations are "static" in the sense that expected income in period t equals actual income in the previous period.

a) Solve for Y_t.

b) Find the income multiplier with respect to a change in $\bar{I}_t$ and ε_t, respectively.

Suppose instead that expectations are rational.

c) Explain what this means.

d) Solve for Y_t.

e) Find the income multiplier with respect to a change in $\bar{I}_t$ and ε_t, respectively.

f) Compare the result under e) with that under b). Comment.

Let us now compare the aggregate consumption function in this simple short-run model with the aggregate consumption function in the Ramsey model with logarithmic instantaneous utility.

g) Write down the latter consumption function.

h) Comment on differences between this and (2).

i) Although (2) is too simple, could there be a good reason for letting the role of expected output in period t enter in a short-run model as it does here? Comment.

F.3 Consider arbitrage between equity shares and a riskless asset paying the constant rate of return $r > 0$. Let p_t denote the price at the beginning of period t of a share that at the end of period t yields the dividend d_t. As seen from period t there is uncertainty about p_{t+i} and d_{t+i} for $i = 1, 2, \ldots$. That is, $E_t d_t = d_t$, $E_t d_{t+i} = d_{t+i} + u_{t+i}$, where u_{t+i} is a random variable. Suppose agents have rational expectations and care only about expected return (risk neutrality).

a) Write down the no-arbitrage condition.

Suppose dividends follow the process $d_t = \bar{d} + \varepsilon_t$, where $\bar{d}$ is a positive constant and ε_t is white noise, observable in period t, but not known in advance.

b) Find the fundamental solution for p_t and let it be denoted p_t^*. *Hint:* given $y_t = aE_t y_{t+1} + c\ x_t$, the fundamental solution is $y_t = cx_t + c\sum_{i=1}^{\infty} a^i E_t x_{t+i}$.

Suppose someone claims that the share price follows the process

$$p_t = p_t^* + b_t,$$

with a given $b_0 > 0$ and, for $t = 0, 1, 2, \ldots$,

$$b_{t+1} = \begin{cases} \frac{1+r}{q_t} b_t & \text{with probability } q_t, \\ 0 & \text{with probability } 1 - q_t, \end{cases}$$

where $q_t = f(b_t)$, $f' < 0$.

c) What is an asset price bubble and what is a rational asset price bubble?

d) Can the described b_t process be a rational asset price bubble? *Hint:* a bubble component associated with the inhomogenous equation $y_t = aE_t y_{t+1} + c\, x_t$ is a solution, different from zero, to the homogeneous equation, $y_t = aE_t y_{t+1}$.

F.4 *Short question*
"Under the hypothesis of rational expectations, speculative bubbles cannot arise in general equilibrium." True or false? Why?

F.5 *The housing market in an old city quarter (partial equilibrium analysis)* Consider the housing market in an old city quarter with unique amenity value (for convenience we will speak of "houses" although perhaps "apartments" would fit real world situations better). Let H be the aggregate stock of houses (apartments), measured in terms of some basic unit (a house of "normal size", somehow adjusted for quality) existing at a given point in time. No new construction is allowed, but repair and maintenance is required by law and so H is constant through time. Notation:

p_t = the real price of a house (stock) at the start of period t,
m = real maintenance costs of a house (assumed constant over time),
$\tilde{R}_t$ = the real rental rate, i.e., the price of housing services (flow),
R_t = $\tilde{R}_t - m =$ the *net* rental rate = net revenue to the owner per unit of housing services.

Let the housing services in period t be called S_t. Note that S_t is a *flow:* so and so many square meter-months are at the disposal for utilization (accommodation) for the owner or tenant during period t. We assume the rate of utilization of the house stock is constant over time. By choosing appropriate measurement units the rate of utilization is normalized to 1, and so $S_t = 1 \cdot H$. The prices p_t, m, and R_t are measured in *real* terms, that is, deflated by the consumer price index. We assume perfect competition in both the market for houses and the market for housing services.

Suppose the aggregate demand for housing services in period t is

$$D(\tilde{R}_t, X_t), \qquad D_1 < 0, D_2 > 0, \tag{*}$$

where the stochastic variable X_t reflects factors that in our partial equilibrium framework are exogenous (for example present value of expected future labor income in the region).

a) Set up an equation expressing equilibrium in the market for housing services. In a diagram in $(H, \tilde{R})$ space, for given X_t, illustrate how $\tilde{R}_t$ is determined.

b) Show that the equilibrium *net* rental rate at time t can be expressed as an implicit function of H, X_t, and m, written $R_t = \mathcal{R}(H, X_t, m)$. Sign the partial derivatives w.r.t. H and m of this function. Comment.

Suppose a constant tax rate $\tau_R \in [0, 1)$ is applied to rental income, after allowance for maintenance costs. In case of an owner-occupied house the owner still has to pay the tax $\tau_R R_t$ out of the imputed income, R_t, per house per year. Assume further there is a constant property tax rate $\tau_p \geq 0$ applied to the market value of houses. Finally, suppose a constant tax rate $\tau_r \in [0, 1)$ applies to interest income, whether positive or negative. We assume capital gains are not taxed and we ignore all complications arising from the fact that most countries have tax systems based on nominal income rather than real income. In a low-inflation world this limitation may not be serious.

We assume housing services are valued independently of whether the occupant owns or rents. We further assume that the market participants are risk-neutral and that transaction costs can be ignored. Then in equilibrium,

$$\frac{(1 - \tau_R)R_t - \tau_p p_t + p_{t+1}^e - p_t}{p_t} = (1 - \tau_r)r, \tag{**}$$

where p_{t+1}^e denotes the expected house price next period as seen from period t, and r is the real interest rate in the loan market. We assume that $r > 0$ and all tax rates are constant over time.

c) Interpret (**).

Assume from now that the market participants have rational expectations (and know the stochastic process which R_t follows as a consequence of the process of X_t).

d) Derive the expectational difference equation in p_t implied by (**).

e) Find the fundamental value of a house, assuming R_t does not grow "too fast". *Hint:* write (**) on the standard form for an expectational difference equation and use that the fundamental solution of the standard equation $y_t = aE_t y_{t+1} + c\, x_t$ is $y_t = cx_t + c \sum_{i=1}^{\infty} a^i E_t x_{t+i}$.

Denote the fundamental value p_t^*. Assume R_t follows the process

$$R_t = \bar{R} + \varepsilon_t, \tag{***}$$

where $\bar{R}$ is a positive constant and ε_t is white noise with variance σ^2.

f) Find p_t^* under these conditions.

g) How does $E_{t-1}p_t^*$ (the conditional expectation one period beforehand of p_t^*) depend on each of the three tax rates? Comment.

h) How does $Var_{t-1}(p_t^*)$ (the conditional variance one period beforehand of p_t^*) depend on each of the three tax rates? Comment.

F.6 *A housing market with bubbles (partial equilibrium analysis)* We consider the same setup as in Exercise F.5, including the equations (*), (**), and (***).

Suppose that until period 0 the houses were owned by the municipality. But in period 0 the houses are sold to the public at market prices. Suppose that by coincidence a large positive realization of ε_0 occurs and that this triggers a stochastic bubble of the form

$$\begin{aligned} b_{t+1} &= [1 + \tau_p + (1 - \tau_r)r]\, b_t + \varepsilon_{t+1}, \qquad t = 0, 1, 2, ..., \quad (1) \\ \text{where} \quad E_t \varepsilon_{t+1} &= 0 \;\text{ and }\; b_0 = \varepsilon_0 > 0. \end{aligned}$$

Until further notice we assume b_0 is large enough relative to the stochastic process $\{\varepsilon_t\}$ to make the probability that b_{t+1} becomes non-positive negligible.

a) Can (1) be a rational bubble? You should answer this in two ways: 1) by using a short argument based on theoretical knowledge, and 2) by directly testing whether the price path $p_t = p_t^* + b_t$ is arbitrage free. Comment.

b) Determine the value of the bubble in period t, assuming ε_{t-i} known for $i = 0, 1, ..., t$.

c) Determine the market price, p_t, and the conditional expectation $E_t p_{t+1}$. Both results will reflect a kind of "overreaction" of the market price to the shock ε_t. In what sense?

d) It may be argued that a stochastic bubble of the described ever-lasting kind does not seem plausible. What kind of arguments could be used to support this view?

e) Still assuming $b_0 > 0$, construct a rational bubble which has a constant probability of bursting in each period $t = 1, 2, ...$

f) What is the expected further duration of the bubble as seen from any period $t = 0, 1, 2, ...$, given $b_t > 0$? *Hint:* $\sum_{i=0}^{\infty} iq^i(1-q) = q/(1-q)$.[1]

g) If the bubble is alive in period t, what is the probability that the bubble is still alive in period $t+s$, where $s = 1, 2, ...$? What is the limit of this probability for $s \to \infty$?

h) Assess this last bubble model.

i) Housing prices are generally considered to be a good indicator of the turning points in business cycles in the sense that house prices tend to move in advance of aggregate economic activity, in the same direction. In the language of business cycle analysts housing prices are a *procyclical leading indicator*. Do you think this last bubble model fits this observation? *Hint:* consider how a rise in p affects residential investment and how this is likely to affect the economy as a whole.

[1] Here is a proof of this formula. $\sum_{i=0}^{\infty} iq^i(1-q) = (1-q)q\sum_{i=0}^{\infty} iq^{i-1} = (1-q)q\sum_{i=0}^{\infty} dq^i/dq = (1-q)qd\left(\sum_{i=0}^{\infty} q^i\right)/dq = (1-q)qd\,(1-q)^{-1}/dq = (1-q)q\,(1-q)^{-2} = q\,(1-q)^{-1}$. $\square$

Appendix A. Solutions to linear differential equations

For a general differential equation of first order, $\dot{x}(t) = \varphi(x(t), t)$, with $x(t_0) = x_{t_0}$ and where φ is a continuous function, we have, at least for t in an interval $(-\varepsilon, +\varepsilon)$ for some $\varepsilon > 0$,

$$x(t) = x_{t_0} + \int_{t_0}^{t} \varphi(x(\tau), \tau) d\tau. \tag{*}$$

To get a confirmation, calculate $\dot{x}(t)$ from (*).

For the special case of a linear differential equation of first order, $\dot{x}(t) + a(t)x(t) = b(t)$, we can specify the solution. Three sub-cases of rising complexity are:

1. $\dot{x}(t) + ax(t) = b$, with $a \neq 0$ and initial condition $x(t_0) = x_{t_0}$. Solution:

$$x(t) = (x_{t_0} - x^*)e^{-a(t-t_0)} + x^*, \text{ where } x^* = \frac{b}{a}.$$

If $a = 0$, we get, directly from (*), the solution $x(t) = x_{t_0} + bt$.[2]

2. $\dot{x}(t) + ax(t) = b(t)$, with initial condition $x(t_0) = x_{t_0}$. Solution:

$$x(t) = x_{t_0}e^{-a(t-t_0)} + e^{-a(t-t_0)} \int_{t_0}^{t} b(s)e^{a(s-t_0)} ds.$$

[2] Some non-linear differential equations can be transformed into this simple case. For simplicity let $t_0 = 0$. Consider the equation $\dot{y}(t) = \alpha y(t)^\beta$, $y_0 > 0$ given, $\alpha \neq 0, \beta \neq 1$ (a Bernoulli equation). To find the solution for $y(t)$, let $x(t) \equiv y(t)^{1-\beta}$. Then, $\dot{x}(t) = (1-\beta)y(t)^{-\beta}\dot{y}(t) = (1-\beta)y(t)^{-\beta}\alpha y(t)^\beta = (1-\beta)\alpha$. The solution for this is $x(t) = x_0 + (1-\beta)\alpha t$, where $x_0 = y_0^{1-\beta}$. Thereby the solution for $y(t)$ is $y(t) = x(t)^{1/(1-\beta)} = \left(y_0^{1-\beta} + (1-\beta)\alpha t\right)^{1/(1-\beta)}$, which is defined for $t > -y_0^{1-\beta}/((1-\beta)\alpha$.

Special case: $b(t) = ce^{ht}$, with $h \neq -a$ and initial condition $x(t_0) = x_{t_0}$. Solution:

$$\begin{aligned} x(t) &= x_{t_0}e^{-a(t-t_0)} + e^{-a(t-t_0)}c\int_{t_0}^{t} e^{(a+h)(s-t_0)}ds \\ &= (x_{t_0} - \frac{c}{a+h})e^{-a(t-t_0)} + \frac{c}{a+h}e^{h(t-t_0)}. \end{aligned}$$

3. $\dot{x}(t) + a(t)x(t) = b(t)$, with initial condition $x(t_0) = x_{t_0}$. Solution:

$$x(t) = x_{t_0}e^{-\int_{t_0}^{t} a(\tau)d\tau} + e^{-\int_{t_0}^{t} a(\tau)d\tau}\int_{t_0}^{t} b(s)e^{\int_{t_0}^{s} a(\tau)d\tau}ds.$$

Special case: $b(t) = 0$. Solution:

$$x(t) = x_{t_0}e^{-\int_{t_0}^{t} a(\tau)d\tau}.$$

Even more special case: $b(t) = 0$ and $a(t) = a$, a constant. Solution:

$$x(t) = x_{t_0}e^{-a(t-t_0)}.$$

Remark 1 For $t_0 = 0$, most of the formulas will look simpler.

Remark 2 To check whether a suggested solution *is* a solution, calculate the time derivative of the suggested solution and add an arbitrary constant. By appropriate adjustment of the constant, the final result should be a replication of the original differential equation together with its initial condition.